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ON THE FOURIER-JACOBI COEFFICIENTS OF CERTAIN EISENSTEIN SERIES ON A UNITARY GROUP

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NOTATION

Throughout this thesis we will use the following notation for the the various completions and adelic groups associated to the rational number field $\mathbf{Q}$, an imaginary quadratic extension $K = \mathbf{Q}(\sqrt{-D})$ of $\mathbf{Q}$ with discriminant $-D$, and an algebraic subgroup of $GL_3(K)$. A complete index to the notation used in this thesis appears after Chapter 8, which lists the notation according to the section in which it is introduced.

Completions, Adeles, and Ideles of the rational number field

$\mathbf{Q}$: the field of rational numbers
$\mathbf{Z}$: the ring of rational integers
$\mathbf{Q}_p$: the field of p-adic numbers
$\mathbf{Q}_p^*$: the multiplicative group of p-adic numbers
$\mathbf{Z}_p$: the ring of p-adic integers
$\mathbf{Z}_p^*$: the group of units of the p-adic integers
$\mathbf{Q}_{\mathbf{A}}$: the adele group of $\mathbf{Q}$
$\mathbf{Q}_{\mathbf{A}}^*$: the idele group of $\mathbf{Q}$
$\mathbf{Q}_\infty \cong \mathbf{R}$: the archimedean completion of $\mathbf{Q}$
$\mathbf{Q}_f$: the group of nonarchimedean adeles
$\mathbf{Q}_f^*$: the group of nonarchimedean ideles

Completions, Adeles, and Ideles of the imaginary quadratic field K

$K = \mathbf{Q}(\sqrt{-D})$: an imaginary quadratic extension of $\mathbf{Q}$ of discriminant $-D$
$\tau = \sqrt{-D}$ generates the relative different of K over $\mathbf{Q}$
$O_K = \mathbf{Z}[(D + \sqrt{-D})/2]$: the ring of integers of K
K_π the completion of K at the prime π
K_π^* the group of units in the ring K_π
O_π the completion of O_K at the prime π
O_π^* the group of units in the ring O_π
$ord_\pi : K_\pi^*/O_\pi^* \to \mathbf{Z}$ the valuation on K_π^*
$K_p \cong K \otimes \mathbf{Q}_p \cong \prod_{\pi|p} K_\pi$
K_p^* the group of units in the ring K_p
$O_p \cong O_K \otimes \mathbf{Z}_p \cong \prod_{\pi|p} O_\pi$
O_p^* the group of units in the ring O_p
$K_\infty \cong K \otimes \mathbf{Q}_\infty \cong \mathbf{C}$ the field of complex numbers
$K_\infty^* \cong \mathbf{C}^*$ the multiplicative group of $\mathbf{C}$
$K_f \cong K \otimes \mathbf{Q}_f$
$K_f^* \cong K \otimes \mathbf{Q}_f^*$

$O_f \cong O_K \otimes \mathbf{Z}_p$
O_f^* the group of units in the ring O_f
$K_{\mathbf{A}} \cong K \otimes \mathbf{Q}_{\mathbf{A}}$ the adele group of K
$K_{\mathbf{A}}^* \cong K \otimes \mathbf{Q}_{\mathbf{A}}^*$ the idele group of K
$I_K \cong K_f^*/O_f^*$: the group of fractional ideals of K
$CL_K \cong K^* \backslash K_{\mathbf{A}}^*/O_f^* K_\infty^*$: the ideal class group of K

Norms, valuations, and characters of K
$\mathcal{N} : I_K \to \mathbf{Q}^*$ the norm map on ideals of K; $\mathcal{N}(a) = [O_K : a]$ for a integral
$\|x\|_p : K_p^*/O_p^* \to \mathbf{R}$ the p-adic norm map defined by $\|x\|_p = \mathcal{N}(x)^{-1}$
$\|x_f\|_f = \prod_p \|x_p\|_p$ the norm map on the finite ideles
$\|x\|_\infty : \mathbf{C}^* \to \mathbf{R}$ the norm map on $\mathbf{C}^*$; $\|x_\infty\| = x_\infty \overline{x}_\infty$
$\|x\| = \|x\|_{\mathbf{A}} = \|x_\infty\|_\infty = \|x_f\|_f$ the adelic norm map: $K^* \backslash K_{\mathbf{A}}^*/O_f^* \to \mathbf{R}^*$
$e(z) = e^{2\pi i z}$ the exponential function
$\lambda : \mathbf{Q} \backslash \mathbf{Q}_A/\mathbf{Z}_f \to \mathbf{C}^*$ the unique continuous character such that $\lambda(t_\infty) = e(t_\infty)$

Algebraic subgroups of $GL_3(K)$ defined over $\mathbf{Q}$
If $H_{\mathbf{Q}}$ is an algebraic subgroup of $Gl_n(K)$ defined over $\mathbf{Q}$, and L is a lattice in K^3, then we will use the following notation to denote the various completions and adelic groups associated to H.
$H_\infty = H_{\mathbf{R}} \subset Gl_n(\mathbf{C})$ the archimedean completion of H.
$H_p = H_{\mathbf{Q}_p}$ the completion of H in $Gl_n(\mathbf{Q}_p)$.
$H(L)$ the stabilizer of L in H
$H(L)_p$ the stabilizer of $L_p = L \otimes \mathbf{Z}_p$ in H_p
$H_{\mathbf{A}}$ the adele group of H
H_f the group of nonarchimedean adeles
$H(L)_f = \prod_p H(L)_p$
We consider the subgroup of rational points $H_{\mathbf{Q}}$ embedded along the diagonal in the adele group $H_{\mathbf{A}}$, and we view H_p, H_f, H_∞ as subgroups of $H_{\mathbf{A}}$ in the usual way.

INTRODUCTION

The purpose of this paper is to develop an explicit formula for the Fourier-Jacobi coefficients of certain adelic Eisenstein series on the non-tube domain $GU(2,1)$ by extending a technique used by Siegel [15] , Baily [1], Tsao [17], and Karel [6] in the context of rational tube domains. This work also relies on results of Shintani [13] concerning the Fourier-Jacobi coefficients of those modular forms on $GU(2,1)$ which are simultaneous eigenforms of the Hecke operators.

Our main theorem provides a formula for the Fourier-Jacobi coefficients of certain Eisenstein series E_Λ of weight k associated to Hecke characters Λ of the imaginary quadratic field K which is used to provide a $\mathbf{Q}$-structure for $GU(2,1)$. The formula is deduced from an expansion of the Eisenstein series which is closely related to the Fourier-Jacobi expansion and which has the form

$$E_\Lambda = \sum_{\xi \in \Xi} L_\xi(k, \Lambda)\ (Z_\xi(k, \Lambda) \cdot \theta_\xi)\,,$$

where the complex number $L_\xi(k,\Lambda)$ is an explicit monomial of Hecke and Dirichlet L-series, $\{\theta_\xi : \xi \in \Xi\}$ is an infinite collection of primitive adelic theta functions, and $Z_\xi(k,\Lambda)$ is a formal Dirichlet series of operators on the graded ring of theta functions which can be expressed in terms of the eigenvalues of E_Λ by a result of Shintani [13]. A consequence of this explicit formula is the arithmeticity (in the sense of Shimura [11]) of the Eisenstein series.

In the remainder of this Introduction, we give the definition of the Eisenstein series that are considered and, after describing Shintani's theory of adelic theta functions, we state the main theorem precisely and sketch its proof.

Let K be an imaginary quadratic extension of $\mathbf{Q}$ with discriminant $-D$, let $R \in GL_3(K)$ be a nondegenerate, indefinite, hermitian matrix, and let $G_{\mathbf{Q}}$ be the $\mathbf{Q}$-algebraic group of similitudes of R; that is the group of all g in $GL_3(K)$ such that ${}^t\overline{g}Rg = \mu(g)R$ for some rational number $\mu(g)$. The $\mathbf{Q}$-isomorphism class of $G_{\mathbf{Q}}$ depends on D but, as will be seen in Chapter 1, it is independent of R. Without loss of generality (Lemma 1.1), we can take for R the following matrix :

$$R = \begin{pmatrix} 0 & 0 & \sqrt{-D} \\ 0 & -D & 0 \\ -\sqrt{-D} & 0 & 0 \end{pmatrix}.$$

The group G_∞ of real points of G acts biholomorphically on the hermitian symmetric domain $\mathcal{D}(R) = \{\xi \in \mathbf{P_C}^2 : {}^t\overline{\xi}R\xi > 0\}$ and the jacobian determinant of the action of $g = (g_{ij})$ at $\xi = (z, w, 1)$ is given by

$$jac(g, \xi) = (g_{31}z + g_{32}w + g_{33})^{-3}det(g).$$

Let P be the parabolic subgroup of G which stabilizes the boundary point ${}^t(1, 0, 0)$. Then P is the semidirect product of a maximal torus D of G and its unipotent radical U. Moreover, $D \cong Z \times T$ where Z is the center of G and both $Z_\mathbf{Q}$ and $T_\mathbf{Q}$ are isomorphic to K^*. Let N denote the center of U, then $N_\mathbf{Q} \cong \mathbf{Q}$ and $U_\mathbf{Q}/N_\mathbf{Q} \cong K$.

Let L be the lattice O_K^3 in K^3. It will be shown that L is a maximal lattice in the sense of Shimura [11], and that the adele group $G_\mathbf{A}$ admits an Iwasawa decomposition, $G_\mathbf{A} = P_\mathbf{A}G(L)_f\mathbf{K}_\infty^0$, where $\mathbf{K}_\infty^0$ is a subgroup of the maximal compact subgroup $\mathbf{K}_\infty$ of G_∞ which stabilizes the point $o = (\sqrt{-D}/2, 0, 1)$ and that $\mathbf{K}_\infty^0 \cap P_\infty$ is a finite subgroup Z_1 of the center of G_∞. Thus, there is a well-defined map (§1.5)

$$\delta : G_\mathbf{A}/G(L)_f\mathbf{K}_\infty^0 \to D_\mathbf{A}/D(L)_fZ_1, \qquad g \in \delta(g)U_\mathbf{A}G(L)_f\mathbf{K}_\infty^0,$$

which will be used to define the Eisenstein series of $G_\mathbf{A}$.

We are concerned here with modular forms on the adele group $G_\mathbf{A}$, which are continuous complex valued functions, F, on $G_\mathbf{Q}\backslash G_\mathbf{A}/G(L)_f$ such that for each fixed $g_f \in G_f$, and an integer $k \geq 0$ called the weight of F, the function $jac(g_\infty, o)^{-k}F(g_\infty g_f)$ depends only on the point $g_\infty o \in \mathcal{D}(R)$ and induces a holomorphic function there. This space of modular forms of weight k is denoted $A_k(L)$. The center of $G_\mathbf{A}$ acts on $A_k(L)$ by translation and the eigencharacters of this representation are ideal class characters. The eigenspace of those modular forms which transform by χ is denoted $A_k(L, \chi)$.

Of particular interest to us are the Eisenstein series, which are modular forms of integer weight, say $k > 0$, associated to Hecke characters Λ of the same weight, k, on the maximal torus of G. These Eisenstein series are defined by the series

$$E_\Lambda(g) = \sum_{\gamma \in P_\mathbf{Q}\backslash G_\mathbf{Q}} \Lambda\delta(\gamma g).$$

The Hecke character Λ is a continuous homomorphism from the group $D_\mathbf{Q}\backslash D_\mathbf{A}/D(L)_f$ to $\mathbf{C}^*$ such that

$$\Lambda(d_\infty d_f) = jac(d_\infty, o)^k\Lambda(d_f).$$

The restrictions of Λ to Z and T are Hecke characters of K, denoted Λ_Z and Λ_T. Λ_Z is a character of the class group, and $\Lambda_T^*(x) = \Lambda_T(x)\|x\|^{3k/2}$ is a unitary Hecke character. The Eisenstein series are simultaneous eigenfunctions of the ring of Hecke operators, and their eigenvalues are computed in Chapter 7. The existence of local Iwasawa decompositions at all places of G implies that E_Λ is determined by its values on $M_\mathbf{A} = T_\mathbf{A}U_\mathbf{A}$.

If F is any modular form in $A_k(L, \chi)$, then its restriction to $M_\mathbf{A}$ admits a Fourier-Jacobi expansion of the form

$$F = \sum_{r \in \mathbf{Q}} F'_r,$$

where the functions F'_r are defined on $M_{\mathbf{A}}$ by

$$F'_r(m) = \int_{N_{\mathbf{A}}} F(nm)\lambda(-r\cdot n)dn.$$

The relationship between these Fourier-Jacobi coefficients and classical theta functions is rather complex. To describe this relationship, we must first partition $M_{\mathbf{A}}$ into countably many open subsets $M_{\mathbf{A}}(c)$,

$$M_{\mathbf{A}}(c) = \{m = ua : u \in U_{\mathbf{A}}, a \in T_{\mathbf{A}}, (a_f\overline{a}_f) = (c)\},$$

indexed by rational numbers c which are norms of ideals of K. Now, let $F'_{r,c}$ denote the restriction of F'_r to $M_{\mathbf{A}}(c)$. In Chapter 3 we will see that $F'_{r,c}$ is non-zero only if rc is a nonnegative integer and that $F'_{r,c}$ can be identified with a tuple of classical theta functions all of which have the same weight which is proportional to rc. Thus, for each rational number r, the Fourier-Jacobi coefficient F'_r is associated to an infinite collection of vectors of classical theta functions of varying weights.

Rather than study these somewhat complicated Fourier-Jacobi coefficients directly (which was Shintani's approach), we investigate a related expansion of F into a series of functions whose relationship with classical theta functions is more direct and simple. For each nonnegative integer ν, let F_ν be the complex valued function on $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}/M(L)_f$ defined by

$$F_\nu(ua) = \int_{N_{\mathbf{A}}} F(nua)\lambda(-\nu\|a_f\|\cdot n)dn.$$

Observe that if $\nu = rc$ then

$$F_\nu|M_{\mathbf{A}}(c) = F'_r|M_{\mathbf{A}}(c).$$

This implies that we obtain an expansion for F in terms of the F_ν:

$$F = \sum_{\nu=0}^{\infty} F_\nu.$$

This series converges uniformly and absolutely on compact subsets of $M_{\mathbf{A}}$. (Indeed, when both sides are restricted to $M_{\mathbf{A}}(c)$ for any fixed c, we obtain the restriction of the Fourier-Jacobi expansion to $M_{\mathbf{A}}(c)$, which converges normally.) To discriminate between this expansion and the Fourier-Jacobi expansion, we will call this expansion the *adelic theta expansion* of F and we call F_ν the νth *adelic theta coefficient.*

The coefficient F_ν belongs to a certain space $V_{k,\nu}(L)$ of adelic theta functions which is isomorphic to a finite direct sum of spaces of classical theta functions of weight proportional to ν. Thus, the functions F_ν can be identified with a vector of theta functions *without having to restrict* to $M_{\mathbf{A}}(c)$.

The space $V_{k,\nu}(L)$ of adelic theta functions consists of all continuous functions θ on the double coset space $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}/M(L)_f$ which satisfy two conditions. First, for any n in the center $N_{\mathbf{A}}$ of the unipotent radical $U_{\mathbf{A}}$ of $M_{\mathbf{A}}$, one has

$$\theta(nua) = \lambda(\nu\|a_f\|\cdot n)\theta(ua),$$

where λ is the unique continuous character on $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}/N(L)_f$ such that under the identification of N_∞ with $\mathbf{R}$, we have $\lambda(n_\infty) = e^{2\pi i n_\infty}$. To describe the second condition we must introduce some additional notation. Let ξ_1, ξ_2 be the complex valued functions defined on M_∞ by

$$m_\infty \cdot o = (\xi_1(m_\infty), \xi_2(m_\infty), 1),$$

where $o = (\sqrt{-D}/2, 0, 1)$ and let $j_k(m_\infty, r) = jac(m_\infty, o)^k\, e(r\xi_1(m_\infty))$. The second condition asserts that for $m = ua \in M_{\mathbf{A}}$ with m_f fixed, the function

$$\tilde{\theta}(m) = j_k(m_\infty, \nu\|a_f\|)^{-1}\, \theta(m),$$

depends only on the point $\xi_2(m_\infty) \in N_\infty\backslash M_\infty/T_\infty \cong \mathbf{C}$ and the function it defines on $\mathbf{C}$ is holomorphic.

In Chapter 3 we show that the space of adelic theta functions is isomorphic to the direct sum of finitely many spaces of classical theta functions, one for each ideal class in K. This isomorphism is made explicit and can be used to construct a basis for the adelic theta functions using Riemann's theta function. There is a natural inner product on this space of theta functions, called the Siegel inner product, which is defined by

$$(\theta, \theta') = \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \overline{\theta(m)}\theta'(m)dm,$$

for some right-invariant Haar measure dm on $M_{\mathbf{A}}$.

Shintani has studied a natural action of the idele group $K^*_{\mathbf{A}}$ on the spaces of adelic theta functions and has developed a theory of "newforms" for the graded ring $\bigoplus_\nu V_{k,\nu}(L)$ with respect to the Siegel inner product, which is similar to theory of newforms in the modular form case.

More precisely, Shintani defines a map, l, from the group I_K of ideals of K to the group of endomorphisms of the graded ring of theta functions. This map has the property that $l(a)$ maps theta functions of weight ν to theta functions of weight $\nu\mathcal{N}(a)$ where $\mathcal{N}(a)$ is the norm of the ideal a. The map l is defined on $T_f/T(L)_f \cong K^*_f/O^*_f \cong I_K$ by

$$\left(l(a)\theta\right)(m) = \int_{U(L)_f} \theta(mua^{-1})du.$$

Observe that the operators $l(a)$ are injections for all integral ideals a. (Indeed, $\theta(mua^{-1}) = \theta(ma)$ for such a). The subspace of imprimitive theta functions is defined to be the space spanned by the images of the maps $l(a)$ where a ranges over all integral ideals not equal to O_K. The orthogonal complement of the space of imprimitive theta functions with respect to the Siegel inner product is called the space of primitive theta functions, and is denoted $V_{k,\nu}(L)^0$. One of Shintani's main theorems asserts that there is an orthogonal decomposition of the space of adelic theta functions into images of the primitive spaces under the maps $l(a)$. More precisely, it is shown that for every positive integer ν, there is a decomposition

$$V_{k,\nu}(L) = \bigoplus_{\substack{a \in I^+ \\ \mathcal{N}(a)|\nu}} l(a)\left(V_{k,\nu/\mathcal{N}(a)}(L)^0\right),$$

where the sum is orthogonal direct and I^+ denotes the monoid of integral ideals of K. Thus, a ranges over the finite set of all integral ideals of K whose norm divides ν. Thus, if $\mathcal{B}_\nu$ is a basis for the space of primitive theta functions of weight ν, and if $\mathcal{B}$ is the union of the $\mathcal{B}_\nu$, then a basis for the ring of adelic theta functions is given by $\{l(a)\theta : a \in I^+, \theta \in \mathcal{B}\}$.

Shintani also shows that the map $a \mapsto \mathcal{N}(a)^{1/2} l(a/\overline{a})$ defines a representation of the group of ideals prime to νD on the space of primitive theta functions of level ν, and hence induces an orthogonal decomposition of the space of primitive theta functions into the eigenspaces, $V_{k,\nu}(L)^0_\kappa$ of this representation, where the eigencharacters κ are essentially Hecke characters. More precisely, if κ is one of the eigencharacters, then

$$\kappa^*(a) = \kappa(a/\overline{a}) \prod_{p\ inert} (-1)^{ord_p(a)}$$

defines a homomorphism on the ideals of K relatively prime to νD which is induced from a Hecke character of conductor dividing $4\nu D$.

These eigenspaces can be further decomposed using a commuting set of projections, l_π, associated to the ramified primes π of K. These projections are diagonalizable endomorphisms with eigenvalues 0 and 1, and are defined by

$$(l_\pi \theta)(m) = \int_{U(L)^*_p} \theta(mu)du,$$

where p is the prime of $\mathbf{Q}$ divisible by π and $U(L)^*_p \supset U(L)_p$ is the subgroup of index p in $U(\pi^{-1}L)_p$ whose intersection with N_p is $N(L)_p$. The simultaneous eigenspaces associated to this set of projections can be classified by subsets Σ of the set of ramified primes for which the eigenvalues are 1. By intersecting these two decompositions, Shintani forms an orthogonal decomposition of the space of primitive theta functions of level ν into eigenspaces $V_{k,\nu}(L)^0_{\kappa,\Sigma}$. (He also shows that when K is the Gaussian number field these eigenspaces have dimension 1 over $\mathbf{C}$, but it is not known whether these eigenspaces have dimension 1 for other fields.)

Shintani's decomposition of the space of primitive theta functions can be used to define a system of coordinates for modular forms which has interesting arithmetic properties. To construct this system of coordinates, Shintani chooses a basis $\mathcal{B}$ of the space of primitive theta functions of all levels, subject to the restriction that all basis elements be in the eigenspaces $V_{k,\nu}(L)^0_{\kappa,\Sigma}$. More precisely, let Ξ_ν be the set of all triples $\xi = (\nu, \kappa^*, \Sigma)$ such that the eigenspace $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$ is nonzero, and let Ξ be the union of the Ξ_ν for all non-negative integers ν. Further, for any $\xi \in \Xi$, let $\mathcal{B}_\xi$ be an orthonormal basis of V_ξ (with respect to the Siegel inner product), and let $\mathcal{B}_\nu$ be the basis of the finite dimensional vector space $V_{k,\nu}(L)^0$ of primitive theta functions of weight ν defined by

$$\mathcal{B}_\nu = \bigcup_{\xi \in \Xi_\nu} \mathcal{B}_\xi.$$

Finally, let $\mathcal{B} = \bigcup_{\nu=0}^\infty \mathcal{B}_\nu$ as before. Once this choice of bases has been made, the adelic theta coefficients F_ν in the expansion

$$F = \sum_{\nu=0}^\infty F_\nu$$

of any modular form F can be written uniquely as a finite linear combination involving translates of elements of $\mathcal{B}$ by Shintani operators; that is,

$$F_\nu = \sum_{\substack{a\in I+\\ \mathcal{N}(a)|\nu}} \sum_{\theta\in\mathcal{B}_{\nu/\mathcal{N}(a)}} z_\theta(a)\,(l(a)\theta)\,,$$

for some complex numbers $z_\theta(a)$ where a ranges over the monoid of integral ideals of K.

This expression can be considerably simplified by viewing the adelic theta expansion to be an element of the topological algebra $\hat{\mathcal{R}}$ whose underlying vector space structure is defined by

$$\hat{\mathcal{R}} = \prod_{\nu=0}^{\infty} V_{k,\nu}(L)$$

with the usual (Tychonoff) product topology, with componentwise addition, and whose multilication is given as follows:

Let $x = (x_0, x_1, \ldots), y = (y_0, y_1, \ldots) \in \hat{\mathcal{R}}$, then $x \cdot y = (z_0, z_1, \ldots)$, where

$$z_n = \sum_{i+j=n} x_i y_j$$

If we associate to each modular form F the element $(F_1, F_2, \ldots)$ in $\hat{\mathcal{R}}$, then we obtain a continuous homomorphism of the graded ring of modular forms into the ring $\hat{\mathcal{R}}$. In this light, the adelic theta expansion for F described above can be written more simply as

$$F = \sum_{\theta\in\mathcal{B}} \sum_{a\in I+} z_\theta(a)\,(l(a)\theta)\,,$$

where for each $\theta \in \mathcal{B}$ the inner sum clearly converges to an element of $\hat{\mathcal{R}}$ and the outer sum over $\mathcal{B}$ also converges in $\hat{\mathcal{R}}$, since only finitely many summands lie in any particular component $V_{k,\nu}(L)$. This expansion can be written more suggestively as

$$F = \sum_{\theta\in\mathcal{B}} (Z_\theta \cdot \theta)\,, \qquad Z_\theta \cdot \theta = \sum_{a\in I+} z_\theta(a)\,(l(a)\theta)\,,$$

where Z_θ is an operator on the ring $\hat{\mathcal{R}}$.

Z_θ can also be viewed as an element of a power series ring $\mathcal{P}$ in infinitely many variables $\{x_\pi\}$ indexed by the prime ideals π of K, with coefficients in the field of complex numbers. These power series operate on the vector space $\hat{\mathcal{R}} = \prod_\nu V_{k,\nu}(L)$ in such a way that the indeterminates x_π act as $l(\pi)$, that is, $x_\pi \cdot \theta = l(\pi)\theta$ for each prime ideal π of K. Indeed, if we let Z_θ be the following element of $\mathcal{P}$

$$Z_\theta = \sum_{a\in I+} z_\theta(a) x_a, \quad \text{where} \quad x_a = \prod_{\pi|a} x_\pi^{ord_\pi(a)},$$

then $Z_\theta \cdot \theta = \sum_a z_\theta(a) l(a)\theta$.

The main theorem of Shintani which we use is a characterization of those modular forms which are simultaneous eigenforms of the Hecke operators in terms of the system of coordinates $\{z_\theta(a) : \theta \in \mathcal{B}, a \in I^+\}$. Shintani's theorem asserts that a modular form $F \in A_k(L, \chi)$ is a simultaneous eigenform of the Hecke operators if and only if for every $\xi \in \Xi$ and every $\theta \in \mathcal{B}_\xi$ the formal Dirichlet series Z_θ is either identically zero or has the form

$$Z_\theta = z_\theta(1) Z_{\xi,\lambda},$$

where $Z_{\xi,\lambda}$ is an explicitly given formal Dirichlet series that depends only on ξ, χ and the eigenvalues λ of F. Moreover, $Z_{\xi,\lambda}$ admits an explicit Euler product factorization of the form:

$$Z_{\xi,\lambda} = \prod_p R_{\xi,\lambda,p},$$

where the Euler factors are explicit rational functions in the indeterminates $\{x_\pi : \pi | p\}$ and are given separately in three cases, depending on whether p is split, inert, or ramified in K.

This theorem of Shintani shows that in order to determine the adelic theta coefficients of a modular form F which is a simultaneous eigenform of the Hecke operators it is neccessary and sufficient to determine the coefficients $\{z_\theta(1) : \theta \in \mathcal{B}\}$ of the primitive components of the adelic theta coefficients and to determine the eigenvalues $\{\lambda(\pi)\}$ of F. This is precisely what is done in this paper for the Eisenstein series associated to unramified Hecke characters of the maximal torus. The method we use is an adaptation of the Siegel-Baily-Tsao-Karel technique for determining the Fourier coefficients of certain Eisenstein series on a tube domain.

Our main theorem asserts, more precisely, that if k is any positive integer and Λ is any unramified Hecke character of weight k on K, and if the Fourier-Jacobi expansion of the Eisenstein series E_Λ is

$$E_\Lambda = \sum_{r \in \mathbf{Q}} E'_{\Lambda,r},$$

where the Fourier-Jacobi coefficients are defined by the integrals

$$E'_{\Lambda,r} = \int_{N_\mathbf{A}} E_\Lambda(nm)\lambda(-r \cdot n)dn,$$

then each Fourier-Jacobi coefficient can be expressed as an infinite sum:

$$E'_{\Lambda,r} = \sum_{c \in \mathcal{N}(I+)} E'_{\Lambda,r} | M_\mathbf{A}(c),$$

where the sets $M_\mathbf{A}(c)$ are all mutually disjoint open sets whose union is all of $M_\mathbf{A}$, and each of the restrictions can be expressed as a finite sum:

$$E'_{\Lambda,r}|M_\mathbf{A}(c) = \sum_{\substack{(\nu,a) \\ \nu a\overline{a} = rc}} \sum_{\xi \in \Xi_\nu} L_\xi(k,\Lambda)\, z_\xi(a)\ (l(a)\theta_\xi)\, |M_\mathbf{A}(c),$$

where ν and a range over integral ideals of $\mathbf{Q}$ and K respectively, and where the theta function $\theta_\xi \in V_\xi$ and the complex numbers $L_\xi(k,\Lambda)$, $z_\xi(a)$ are given explicitly in terms k, ξ, a, and Λ. More precisely, the number $L_\xi(k,\Lambda)$ is a monomial of special values of Hecke and Dirichlet L-series and and the complex numbers $\{z_\xi(a) : a \in I^+\}$ are the coefficients of the formal powerseries

$$Z_\xi(k,\Lambda) = \sum_{a \in I^+} z_\xi(a) x_a,$$

which is given explicitly below as an Euler product.

The main formula for $E'_{\Lambda,r}$ just described is derived from a formula for the *adelic theta coefficients* $E_{\Lambda,\nu}$ of E_Λ:

$$E_{\Lambda,\nu} = \sum_{\substack{(\nu',a) \\ \nu'\mathcal{N}(a)=\nu}} \sum_{\xi \in \Xi_{\nu'}} L_\xi(k,\Lambda)\, z_\xi(a)\, \left(l(a)\theta_\xi\right),$$

which can be expressed more elegantly when the adelic theta expansion of E_Λ is viewed as an element of $\hat{\mathcal{R}}$ as follows:

$$E_\Lambda = \sum_{\xi \in \mathcal{B}} L_\xi(k,\Lambda) \left(Z_\xi(k,\Lambda) \cdot \theta_\xi\right).$$

The constant $L_\xi(k,\Lambda)$, the formal Dirichlet series $Z_\xi(k,\Lambda)$, and the theta function θ_ξ are given explicitly in terms of $\xi = (\nu, \kappa^*, \Sigma)$ as follows:

i) The adelic theta function θ_ξ is defined to be the element of $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$ dual (with respect to the Siegel inner product) to the linear functional l_Λ defined by

$$l_\Lambda(\theta) = \sum_{\beta \in CL_K} \Lambda(\beta_f)^{-1}\ e(-\nu\|\beta_f\|\sqrt{-D}/2)\ \theta(\beta_f),$$

where the summand is a well-defined function on the group $K^*\backslash K^*_{\mathbf{A}}/O^*_f K^*_\infty$ which is isomorphic to the class group CL_K. In other words, θ_ξ is the unique element in V_ξ such that $(\theta_\xi, \theta) = l_\Lambda(\theta)$ for all $\theta \in V_\xi$.

ii) the constant $L_\xi(k,\Lambda)$ is given explicitly as the following monomial of Hecke and Dirichlet L-series:

$$L_\xi(k,\Lambda) = \frac{(-1)^{3k/2}}{4\nu^2 D^{(3k+1)/2}}\ \alpha(k,\Lambda,\Sigma)\ \frac{L_K((3k-1)/2, \chi_2^*)}{L_{\mathbf{Q}}(3k-1,\chi_K)\ L_K(3k/2,\chi_1^*)},$$

where

$$\alpha(k,\Lambda,\Sigma) = \prod_{\pi \in \Sigma} \left(1 + \Lambda_T^*(\pi)\mathcal{N}(\pi)^{-(3k/2)+1}\right),$$

and where χ_1^*, χ_2^* are Hecke characters of K (cf. §1.6) associated to the Hecke characters Λ and κ^* by

$$\chi_1^* = \overline{\Lambda_Z^2\Lambda_T^{*3}} \in \mathcal{H}^*_{-3k,1}(K), \qquad \chi_2^* = \overline{\Lambda_Z^2\Lambda_T^*}\kappa^* \in \mathcal{H}^*_{-(3k-1),C_\nu}(K).$$

iii) The formal Dirichlet series $Z_\xi(k, \Lambda)$ is given explicitly in §7.1. The formula there is derived using Shintani's general formula and our calculation of the eigenvalues of the Eisenstein series.

From this explicit formula, and an explicit isomorphism between the space of adelic theta functions and a space of classical theta functions, we easily deduce the arithmeticity of E_Λ using results of Siegel [14] on the special values of Dirichlet L-series and results of Damerell [4] on the special values of Hecke L-series.

Our formula is proved by extending ideas used by Siegel, Baily, and Tsao, to find the Fourier coefficients of modular forms on tube domains. The situation is, of course, more complicated here because the Fourier-Jacobi coefficients are sums of vectors of theta functions rather than complex numbers.

The first step in the proof is to use the Bruhat decomposition to express the Eisenstein series as a sum over $U_{\mathbf{Q}}$

$$E_\Lambda(g) = \Lambda(\delta(g)) + \sum_{u \in U_{\mathbf{Q}}} \Lambda(\delta(\iota u g)),$$

where ι is a representative of the nontrivial element in the Weyl group. From this representation, we can express the νth adelic theta coefficient of E_Λ (restricted to $M_{\mathbf{A}}$) in the form:

$$E_{\Lambda,\nu}(m) = \sum_{u \in N_{\mathbf{Q}} \backslash U_{\mathbf{Q}}} S_\Lambda(um, \nu \|\delta(m_f)\|),$$

where the summand is defined by an integral of the form

$$S_\Lambda(m, r) = \int_{N_{\mathbf{A}}} \Lambda(\delta(\iota n m)) \lambda(-r \cdot n) dn,$$

and λ is the continuous character on $N_{\mathbf{Q}} \backslash N_{\mathbf{A}} / N(L)_f$ which is used to define the adelic theta coefficients.

The function S_Λ plays a major role in our work and we call it the Siegel function. In Chapter 5 we explicitly evaluate this function and show that, for each fixed $m \in M_{\mathbf{A}}$, it admits a factorization into Euler factors. This evaluation is done by explicitly calculating the Iwasawa map δ and using these calculations to explicitly evaluate the Siegel function. For example, if $m \in M_{\mathbf{A}}$ and $r \in \mathbf{Q}_{\mathbf{A}}$ then we show that

$$S_\Lambda(m, r) = \frac{(-2\pi i)^{3k}}{(3k-1)!} r_\infty^{3k-1} j_k(m_\infty, r_\infty) \prod_p S_{\Lambda,p}(m_p, r_p),$$

where $S_{\Lambda,p}(m_p, r_p) = (1 - p^{-3k})$ if r_p is a unit in $\mathbf{Z}_p$ and $m_p \in M(L)_p$. The restriction of the Siegel function to $M_f \times \mathbf{Z}_f^*$ does not however have compact support and this makes for interesting problems later on in the proof.

In Chapter 6, we show that the primitive component of $E_{\Lambda,\nu}$ is a finite sum

$$E^0_{\Lambda,\nu} = \sum_{\xi \in \Xi_\nu} L_\xi(k, \Lambda) \cdot \theta_\xi,$$

where θ_ξ is the element of V_ξ dual to the linear functional l_Λ defined above. Since the subspaces V_ξ are all orthogonal to each other, we can prove this formula by calculating the inner product of any $\theta \in V_\xi$ with $E_{\Lambda,\nu}$ and showing that

$$(E_{\Lambda,\nu}, \theta) = \overline{L_\xi(k,\Lambda)} l_\Lambda(\theta),$$

where the inner product is defined by the integral

$$(E_{\Lambda,\nu}, \theta) = \int_{M_{\mathbf{Q}} \backslash M_{\mathbf{A}}} \overline{E_{\Lambda,\nu}(m)} \theta(m) dm,$$

with respect to a suitably normalized Haar measure dm. Using the Siegel-Baily-Tsao-Karel form of the Eisenstein series, this inner product can be written in terms of the Siegel function as

$$(E_{\Lambda,\nu}, \theta) = \int_{N_{\mathbf{Q}} T_{\mathbf{Q}} \backslash M_{\mathbf{A}}} \overline{S_\Lambda(m,\nu)} \theta(m) dm.$$

The evaluation of this inner product is done in three steps.

In the first step, we evaluate the archimedean contribution to this integral and show that if θ is any theta function of level ν, then the inner product in question can be expressed as an integral that involves only the nonarchimedean values of the Siegel function and a transform of the theta function:

$$(E_{\Lambda,\nu}, \theta) = \epsilon_\infty \int_{J_K \times W_f} \mathcal{S}_{\Lambda,\theta}(a,w) da dw,$$

where the integrand,

$$\mathcal{S}_{\Lambda,\theta}(a,w) = \Lambda(a_f)^{-1} \overline{S_{\Lambda,f}(w, ra\overline{a}/\mathcal{N}(a))} \tilde{\theta}(aw),$$

involves the finite component of the Siegel function S_Λ and where J_K is the idele class group and $W_f \cong O_f$ is the quotient of the unipotent group U by its center, ϵ_∞ is an explicit algebraic number, and $\tilde{\theta}$ is a simple transform of the theta function.

In the rest of Chapter 6 we evaluate this integral by local methods. Since the integrand does not factor into a product of local functions, we must use a limiting argument. The main idea is to show that for every prime p there is a relation of the form

$$\int_{J_K \times W_p} \mathcal{S}_{\Lambda,\theta}(a, v + w_p) da dw_p = \epsilon_p \int_{J_K} \mathcal{S}_{\Lambda,\theta}(a,v) da,$$

where ϵ_p is an explicit Euler factor and v is any adele whose p-component is zero. To prove formulas of this form we need to assume that θ is in one of Shintani's eigenspaces $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$, and we need to use our explicit evaluation of the Siegel function.

Finally, in Chapter 7 we calculate the eigenvalues of the Eisenstein series and gather together the results of the previous Chapters into our main theorem. In Chapter 8, we deduce the arithmeticity of the Eisenstein series as an easy corollary of the main theorem.

CHAPTER 1

THE ARITHMETIC OF $GU(2,1)$

In this section we review some of the known results about the arithmetic of $GU(2,1)$ including the Bruhat and Iwasawa decompositions, and the class number. We also introduce most of the general notation that will be used in the sequel.

1.1. The $\mathbf{Q}$*-structure on* $GU(2,1)$

Let R be a nondegenerate, indefinite, hermitian form of rank 3 defined over an imaginary quadratic extension K of $\mathbf{Q}$, and let $-D$ be the discriminant of K, $\tau = \sqrt{-D}$, and O_K the ring of integers of K. The reductive $\mathbf{Q}$-algebraic group of similitudes of R is

$$GU(R)_{\mathbf{Q}} = \left\{g \in GL_3(K) : {}^t\overline{g}Rg = \mu(g)R\right\}.$$

A well-known result of Landherr [7] asserts that any indefinite Hermitian form R of rank 3 defined over K can, by appropriate choice of basis of K^3 be put in the form:

$$R_x = \sqrt{-D}\begin{pmatrix} 0 & 0 & 1 \\ 0 & x & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where $x \in K^*$, and $\overline{x} = -x$. and that the K-isomorphism class of the inner product space (K^3, R) is classified by the image of $N^K_{\mathbf{Q}}(x) = x\overline{x}$ in $\mathbf{Q}^*/\mathbf{Q}^{*2}$.

Lemma 1.1. *$GU(R) \cong GU(R_\tau)$ as $\mathbf{Q}$-algebraic groups.*

Proof: Without loss of generality, we may assume that R has the form R_x. Let δ be the diagonal matrix: $diag(1,1,m^{-1})$, for $m = x/\tau$ in $\mathbf{Q}$, and observe that ${}^t\overline{\delta}R_\tau\delta = m^{-1}R_{m\tau}$. Thus $GU(R_{m\tau}) = \delta^{-1}GU(R_\tau)\delta$, and $\gamma \mapsto \delta\gamma\delta^{-1}$ is a $\mathbf{Q}$-isomorphism. **Q.E.D.**

By the lemma, we need only consider the group $G = GU(R_\tau)$, where $\tau = \sqrt{-D}$, and $-D$ is the discriminant of K over $\mathbf{Q}$, and we will simply write R for R_τ, where

$$R = \begin{pmatrix} 0 & 0 & \sqrt{-D} \\ 0 & -D & 0 \\ -\sqrt{-D} & 0 & 0 \end{pmatrix}.$$

1.2. The Bruhat decomposition

Let P be the parabolic subgroup of G fixing the point ${}^t(1,0,0)$. Then P is the semidirect product of its unipotent radical U and a maximal torus D of G. Let Z denote the center of G, then $D = Z \times T$ where both Z and T are isomorphic to $R^K_{\mathbf{Q}}(\mathbf{G_m})$. where $\mathbf{G_m}$ is the multiplicative group, and $R^K_{\mathbf{Q}}$ is Weil's groundfield reduction functor.

This isomorphism is realized by the map $d : Z \times T \mapsto D$ defined by

$$d(z,x) = diag(z\overline{x}, z, z/x) = \begin{pmatrix} z\overline{x} & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & zx^{-1} \end{pmatrix}.$$

The unipotent group U is a two stage nilpotent group. Let N be the center of U, and let $W = U/N$, then $N \cong \mathbf{G_a}$ and $W \cong R^K_{\mathbf{Q}}(\mathbf{G_a})$, where $\mathbf{G_a}$ is the additive group, and $R^K_{\mathbf{Q}}$ is Weil's groundfield reduction functor. A short calculation reveals that $U = \{[w,n] : w \in R^K_{\mathbf{Q}}(\mathbf{G_a}), n \in \mathbf{G_a}\}$ where

$$[w,n] = \begin{pmatrix} 1 & \tau\overline{w} & n + w\overline{w}\tau/2 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the group law is given, for any $w_1, w_2 \in R^K_{\mathbf{Q}}(\mathbf{G_a})$, by

$$[w_1,n_1][w_2,n_2] = [w_1 + w_2, n_1 + n_2 + (\overline{w}_1 w_2 - w_1\overline{w}_2)\tau/2),$$

and for any $z, x \in R^K_{\mathbf{Q}}(\mathbf{G_m})$,

$$d(z,x)[w,n]d(z,x)^{-1} = [xw, x\overline{x}n].$$

The solvable subgroup $M = TU$ will play a major role later, as we will be able to restrict modular forms to this subgroup without losing any information. We will denote elements of M by $a[w,t]$ or $[w,t]a$.

Lemma 1.2. *[Bruhat Decomposition] Let D_s denote the maximal $\mathbf{Q}$-split torus of G contained in D. Then D_s is its own centralizer and its normalizer is generated by D_s and the element $\iota \in G_{\mathbf{Q}}$, defined by*

$$\iota = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Moreover, $G_{\mathbf{Q}} = P_{\mathbf{Q}} \cup U_{\mathbf{Q}} D_{\mathbf{Q}} \iota U_{\mathbf{Q}}$ and the map $(u_1, d, u_2) \mapsto u_1 d \iota u_2$ establishes a bijection of $U_{\mathbf{Q}} \times D_{\mathbf{Q}} \times U_{\mathbf{Q}}$ onto $G_{\mathbf{Q}} - P_{\mathbf{Q}}$.

Proof: Since $D_s = \{d(z,x) : z, x \in \mathbf{Q}^*\}$ it is easy to verify that D_s is its own centralizer and that its normalizer is generated by ι and D_s. The second part is simply the specialization of the Bruhat decomposition [3] to this case. **Q.E.D.**

1.3. The action on a hermitian symmetric domain

The group G_∞ acts biholomorphically on $\mathbf{P}^2_{\mathbf{C}}$ by right multiplication, and preserves the hermitian symmetric space $D(R)$ defined by

$$\mathcal{D}(R) = \{\xi \in \mathbf{P}^2_{\mathbf{C}} : {}^t\overline{\xi}R\xi > 0\} = \left\{(z, w, 1) : 2\,Im(z) > |\sqrt{D}|\,|w|^2\right\}.$$

The homomorphism of G_∞ into the group of biholomorphic automorphisms of $D(R)$ is surjective, and the kernel is the center Z_∞ of G_∞. Let $jac(g,\xi)$ be the jacobian determinant of the action of g at the point ξ. By using the Bruhat decomposition

$$G_\infty = D_\infty U_\infty \iota U_\infty \cup D_\infty U_\infty,$$

, it is easy to check that if $g=(g_{ij}) \in G_\infty$, then

$$jac(g,(z,w,1)) = det(g)\ (g_{31}z+g_{32}w+g_{33})^{-3}.$$

1.4. The class number

The class number of unitary groups with respect to certain (maximal) lattices has been calculated by Shimura [10]. In this section we describe those results of his which we will need in our investigation of Eisenstein series.

Let L be an O_K-lattice in K^3. The norm $\mu(L)$ of L is the fractional ideal of K generated by the norms ${}^t\overline{\lambda}R\lambda$ of all elements $\lambda \in L$. An O_K-lattice L is maximal if it is maximal among all O_K-lattices of the same norm. If L_1 and L_2 are two O_K-lattices of K^3, then $[L_1/L_2]$ denotes the ideal generated by $\{det(\alpha)\}$ where α ranges over all elements of $M_3(K)$ such that $\alpha L_1 \subseteq L_2$. Let L_p be the closure of L in K_p^3, and let L_f be the product of the L_p over all rational primes p. If $g \in G_{\mathbf{A}}$, let gL denote the lattice in K^3 such that for all primes p, $(gL)_p = g_pL_p$. Shimura established a local criterion for the maximality of an O_K lattice [loc. cit., Prop. 3.2, 4.7]) which allows one to verify that O_K^3 is a maximal O_K-lattice. He also calculates the class number of a genus of maximal O_K-lattices ([loc. cit.,Thm. 5.24.(ii)]). From his proof of this class number calculation we obtain the following more detailed result:

Proposition 1.4. *[Shimura] Let L be a maximal O_K-lattice in K^3, and for $g \in G_{\mathbf{A}}$, let $\phi(g) = \mu(L)\mu(gL)^{-1}[L/gL]$, then ϕ induces a bijection between the double cosets $G_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_fG_\infty$ and the group of ideal classes CL_K of K.*

Proof: A corollary of the proposition is that the class number of L is equal to the class number of K, and Shimura states this as Theorem 5.24.(ii) of [10]. He actually proves the more detailed result which we have called Proposition 1.4. His proof can be found in [10, p.400, lines 13-23]. **Q.E.D.**

As a simple application of this result, we show that each of the double cosets in Proposition 1.4 has a representative in $D_{\mathbf{A}}$. By [loc. cit., Prop. 2.12, §2.15, Prop. 5.5)] one sees that ϕ restricted to $D_{\mathbf{A}}$ is given by $d \mapsto \mu(d)^{-1}det(d)$, and so induces a map

$$D_{\mathbf{Q}}\backslash D_{\mathbf{A}}/D(L)_fD_\infty \to K^*\backslash K^*_{\mathbf{A}}/O_f^*K^*_\infty\ (\cong CL_K),$$

which takes $d(z,x) \in D_{\mathbf{A}}$ to the ideal class of z^3x^{-2}. Since this is a surjective map, each of the double cosets in Proposition 1.4 has a representative in $D_{\mathbf{A}}$, as was claimed.

Proposition 1.4 has been included only to show that an explicit set of representatives for the double cosets

$$G_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_fG_\infty$$

can easily be constructed. We do not use Proposition 1.4 in the sequel.

Moreover, throughout the remainder of this paper, we will consider only the lattice O_K^3 which will be denoted L.

1.5. The local Iwasawa decompositions

Let $\mathbf{K}_\infty$ be the maximal compact subgroup of G_∞ which stabilizes the point $o = (\tau/2, 0, 1)$. Since P_∞ acts transitively on the hermitian symmetric domain $D(R)$, we have an Iwasawa decomposition at the infinite place: $G_\infty = P_\infty \mathbf{K}_\infty$, where

$$P_\infty \cap \mathbf{K}_\infty = \{d(z,x) : \|z\| = \|x\| = 1\}.$$

Let $\mathbf{K}_\infty^0$ be the intersection of the kernels of the homomorphisms $k \mapsto det(k)$ and $k \mapsto jac(k,o)$ restricted to $\mathbf{K}_\infty$. Since $det(d(z,x)) = z^3\overline{x}/x$ and $jac(d(z,x),o) = x^2\overline{x}$, we see that $P_\infty \cap \mathbf{K}_\infty^0 = Z_1 = \left\{d(\zeta,1) : \zeta^3 = 1\right\}$ and we still have an Iwasawa decomposition: $G_\infty = P_\infty \mathbf{K}_\infty^0$.

In Chapter 5 we will prove that $G_p = P_p G(L)_p$ for each rational prime p. Assuming these decompositions, let $\mathbf{K}_\mathbf{A}^0$ be the compact subgroup of $G_\mathbf{A}$ defined by

$$\mathbf{K}_\mathbf{A}^0 = \mathbf{K}_\infty^0 \prod_p G(L)_p.$$

Then we have an Iwasawa decomposition $G_\mathbf{A} = P_\mathbf{A}\mathbf{K}_\mathbf{A}^0$, and hence there is a map

$$\delta : G_\mathbf{A}/\mathbf{K}_\mathbf{A}^0 \to D_\mathbf{A}/D(L)_f Z_1,$$

defined by $g \in \delta(g)U_\mathbf{A}\mathbf{K}_\mathbf{A}^0$. The most important properties of this map, for our purposes, are (1) if $p \in P_\mathbf{A}$, then $\delta(pg) = \delta(p)\delta(g)$, (2) if $g \in G_\mathbf{A}$, then

$$\delta(P_\mathbf{Q} g G(L)_f) \subset D_\mathbf{Q}\delta(g)D(L)_f,$$

and (3) for any $g \in G_\infty$, $jac(g_\infty, o) = jac(\delta(g_\infty), o)$. This map is used in §2 to associate Eisenstein series to Hecke characters.

1.6. Hecke characters

The role of Dirichlet characters in the work of Siegel, Baily, Tsao, and Karel will be played in our case by Hecke characters.

Let k be an integer divisible by the order, k_0, of the group of units,O_K^*, of K. The set, $\mathcal{H}_k^*(K)$, of unitary, unramified Hecke characters, χ, of K of weight k, consists of all continuous homomorphisms

$$\chi : K^*\backslash K_\mathbf{A}^*/O_f^* \to \{z \in \mathbf{C} : |z| = 1\},$$

such that $\chi(x_\infty) = (x_\infty/|x_\infty|)^k$. Notice that the value of an unramified Hecke character at a nonarchimedean idele $x_f \in K_f^*$ depends only on the ideal associated to x_f. Thus, an unramified Hecke character χ induces a map, say $\hat{\chi}$, on the group I_K of ideals of K such that if (α) is any principal ideal of K, then

$$\hat{\chi}((\alpha)) = (\alpha/|\alpha|)^k, \qquad \alpha \in K^*.$$

Conversely, any such homomorphism $\hat{\chi}$ of I_K induces a Hecke character on $K^*_{\mathbf{A}}$.

The unitary, unramified Hecke characters of weight $k = 0$ are the characters of the class group

$$CL_K = K^* \backslash K^*_{\mathbf{A}} / O^*_f K^*_\infty$$

of K. It is not difficult to see that the set of unitary, unramified Hecke characters of weight k are in 1-1 correspondence with $\mathcal{H}^*_0(K)$. Indeed, if $\chi_1, \chi_2 \in \mathcal{H}^*_k(K)$, then $\chi_1 \chi_2^{-1} \in CL^*_K$. Conversely, suppose that $k_0|k$; it will suffice to construct a single element of $\mathcal{H}^*_k(K)$. Let $\mathcal{E}_k(K)$ denote the vector space of continuous complex valued functions, ϕ, on the idele class group

$$J = K^* \backslash K^*_{\mathbf{A}} / O^*_f$$

such that for all $x \in K^*_{\mathbf{A}}$ we have

$$\phi(x) = (x_\infty/|x_\infty|)^k \, \phi(x_f)$$

Since J is isomorphic to $CL_K \times \mathbf{C}^*/O^*_K$, and since $k_0|k$, it is clear that $\mathcal{E}_k(K)$ is a complex vector space of dimension equal to the class number h of K. Observe that J acts on $\mathcal{E}_k(K)$ by translation:

$$(x \cdot \phi)(y) = \phi(xy), \qquad \forall x, y \in J, \ \phi \in \mathcal{E}_k(K).$$

It will suffice to show that there is some $\phi \in \mathcal{E}_k(K)$ such that for all $x \in K^*_{\mathbf{A}}$, $x \cdot \phi = \alpha(x)\phi$ for some scalar $\alpha(x) \in K^*$, since the function α will then be a Hecke character of weight k for K. Observe that any $x \in K^*_\infty K^* O^*_f$ acts by scalar multiplication on $\mathcal{E}_k(K)$. Indeed, for any $\phi \in \mathcal{E}_k(K)$ we have

$$(zx\omega \cdot \phi)(y) = (z/|z|)^k \, \phi(y), \forall z \in K^*_\infty, x \in K^*, \omega \in O^*_f$$

Let $\zeta : J \to PGL(\mathcal{E}_k(K))$ be the projective representation induced by the action of J on $\mathcal{E}_k(K)$. Then, the kernel of J contains $K^*_\infty K^* O^*_f$. Moreover, ζ is the projective representation on J induced by the regular representation of CL_K. Since the regular representation of CL_K decomposes as a direct sum of characters, $zeta$ admits a fixed point. Thus there exists a Hecke character of weight k. From the previous discussion, it follows that the set of Hecke characters of all integral weights

$$\bigcup_{k_0|k} \mathcal{H}^*_k(K)$$

forms a group isomorphic to $\mathcal{H}^*_0(K) \times \mathbf{Z}$, where the group operation is pointwise multiplication.

Hecke characters with conductor are defined by relaxing the condition that $\chi|O^*_f$ be trivial and requiring instead that its kernel have finite index in O^*_f. More precisely, the set, $\mathcal{H}^*_{k,C}(K)$, of Hecke characters of weight k and conductor C consists of all continous homomorphisms, χ, from $K^*\backslash K^*_{\mathbf{A}}$ into $\mathbf{C}^*$, such that $\chi(x_\infty) = (x_\infty/|x_\infty|)^k$ and such that for all finite primes π of K we have

$$ker(\chi|K^*_\pi) \supset \begin{cases} 1 + \pi^{ord_\pi(C)}O_\pi & \text{if } \pi|C \\ O^*_\pi & \text{if } \pi \not| C \end{cases}.$$

The unramified Hecke characters are those with conductor O_K and the Hecke characters with weight 0 and conductor dividing C form a group which is isomorphic to the character group of the generalized ideal class group associated to C [8].

To each Hecke character of K, we can associate in the standard way an L-series, by using the Hecke character to define a map from the ideals of K to $\mathbf{C}^*$. If the Hecke character χ has conductor C, then it defines a homomorphism from the ideals prime to C into $\mathbf{C}^*$. We extend this to a map of the entire group of ideals by setting $\chi(\pi) = 0$ for all prime ideals π that divide C. Special values of such L-series will arise in the formula for the Fourier-Jacobi coefficients of Eisenstein series. We recall the definition of these L-series for later use.

Let F be a finite field extension of $\mathbf{Q}$, and let χ be a homomorphism from the group of ideals of F into $\mathbf{C}^*$. If p is a prime of $\mathbf{Q}$, we define the local L-factor $\epsilon_{F,p}(s,\chi)$ of χ at p by

$$\epsilon_{F,p}(s,\chi) = \prod_{\pi|p} \left(1 - \chi(\pi) N^F_{\mathbf{Q}}(\pi)^{-s}\right)^{-1}.$$

The L-series associated to χ is then the product of these local L-factors:

$$L_F(s,\chi) = \prod_p \epsilon_{F,p}(s,\chi).$$

The arithmetic properties of special values of these L-series have been investigated by Damerell in the case of Hecke characters and Siegel in the case of Dirichlet characters. We will use their results in Chapter 9 when we deduce the arithmeticity of Eisenstein series as an easy Corollary of our main theorem.

1.7. Hecke characters on the maximal torus

Eisenstein series of $G_{\mathbf{Q}}$ will be constructed from "Hecke characters" of the maximal torus D of $G_{\mathbf{Q}}$ which are lifted to $G_{\mathbf{A}}$ using the function δ of §1.5. We complete this chapter with a discussion of such characters and their lifts.

Let $\mathcal{E}_k(D)$ be the complex vector space of continuous, complex–valued functions

$$\Phi : D_{\mathbf{Q}}\backslash D_{\mathbf{A}}/D(L)_f Z_\infty \to \mathbf{C}$$

such that $\Phi(d_\infty d_f) = jac(d_\infty, o)^k \Phi(d_f)$. If we let $\mathcal{H}_k(D)$ be the subset of $\mathcal{E}_k(D)$ consisting of those Φ which are multiplicative homomorphisms into $\mathbf{C}^*$, then $\mathcal{H}_k(D)$ is a basis for $\mathcal{E}_k(D)$.

Recall that $D_{\mathbf{Q}}$ is isomorphic to K^{*2} via the map $d : K^{*2} \cong D_{\mathbf{Q}}$ (§1.2). It is easy to see that d induces an isomorphism

$$d^* : \mathcal{H}_k(D) \to \mathcal{H}_0^*(K) \times \mathcal{H}_k^*(K) \qquad \Lambda \mapsto \Lambda_Z \Lambda_T^*.$$

Indeed, for $\Lambda \in \mathcal{H}_k(D)$, let

$$\Lambda_Z(z) = \Lambda(d(z,1)) \quad \text{and} \quad \Lambda_T(x) = \Lambda(d(1,x)).$$

Then $\Lambda_Z \in \mathcal{H}_0^*(K)$ is an ideal class character and Λ_T induces a continuous homomorphism from $K^*\backslash K_{\mathbf{A}}^*/O_f^*$ into $\mathbf{C}^*$ such that

$$\Lambda_T(x_\infty) = \left(x_\infty^2 \overline{x}_\infty\right)^k = (x_\infty/|x_\infty|)^k \, \|x_\infty\|^{3k/2}.$$

Thus, if we let $\Lambda_T^*(x) = \|x\|^{-3k/2}\Lambda_T(x)$, then $\Lambda_T^* \in \mathcal{H}_k^*(K)$ is an unramified, unitary Hecke character of weight k, and

$$\Lambda(d(z,x)) = \Lambda_Z(z)\Lambda_T^*(x)\|x_\infty\|^{3k/2}\mathcal{N}(x_f)^{-3k/2}.$$

We call $\mathcal{H}_k(D)$ the set of Hecke characters of weight k on D. The set $\mathcal{H}_k(T)$ of Hecke characters of weight k on T is defined similarly.

Observe that if π is a prime of K dividing a prime p of $\mathbf{Q}$, then $\Lambda_T^*(\pi)$ is 1 if π is inert in K. If π is ramified, then π^2 is principal and so $\Lambda_T^*(\pi) = \pm 1$. If p splits as $\pi\overline{\pi}$ and if h is the class number of K, then π^h is a principal ideal (α) and so $\Lambda_T(\pi)^h = (\alpha/\overline{\alpha})^k$. Similarly, $\Lambda_Z(\pi)$ is 1 if π is inert and is ± 1 if π is ramified. When π splits, $\Lambda_Z(\pi)$ is an hth root of unity.

1.8. Lifting Hecke characters

Let $\mathcal{E}_k(G)$ be the vector space of complex–valued functions Φ on $P_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_f$ such that $\Phi(g_\infty g_f) = jac(g_\infty, o)^k \Phi(g_f)$ for all $g \in G_{\mathbf{A}}$. We will see later that this vector space is isomorphic to the vector space of Eisenstein series of weight k on $G_{\mathbf{A}}$ with respect to $G(L)$, provided k is large enough to guarantee convergence of the Eisenstein series. The next lemma shows that a basis for $\mathcal{E}_k(G)$ can be formed by lifting Hecke characters on D. This will allow us to consider in Chapter 2 only those Eisenstein series constructed from Hecke characters.

Lemma 1.8. *The restriction to $D_{\mathbf{A}}$ induces an isomorphism of $\mathcal{E}_k(G)$ onto the vector space $\mathcal{E}_k(D)$. The inverse map ψ is given by $\psi(\Lambda) = \Lambda \circ \delta$.*

Proof: First we show that $\mathcal{E}_k(D)$ is isomorphic to the vector space $C(B_D)$ of continuous complex valued functions on the finite set $B_D = D_{\mathbf{Q}}\backslash D_{\mathbf{A}}/D(L)_f D_\infty$. Let $\{d_b : b \in B_D\}$ be a complete set of representatives for these double cosets. For example, if $\{\zeta_i : i = 1, \ldots, h\}$ is a complete set of ideal class representatives for K, then $\{d(\zeta_i, \zeta_j) : i, j = 1, \ldots, h\}$ is a complete set of representatives for the double cosets constituting B_D. Since $D_{\mathbf{A}} = \bigcup_b D_{\mathbf{Q}} d_b D(L)_f D_\infty$, any function h on B_D can be uniquely converted to an element g of $\mathcal{E}_k(D)$ by defining

$$g(D_{\mathbf{Q}} d_b D(L)_f d_\infty) = h(d_b) jac(d_\infty, o)^k.$$

Similarly, the vector space $\mathcal{E}_k(G)$ is isomorphic to the space $C(B_G)$ of continuous complex valued functions on the finite set $B_G = P_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_f G_\infty$.

Next, we show that ψ is an injective vector space homomorphism. Observe that if $p \in P_{\mathbf{Q}}$, $g \in G_{\mathbf{A}}$, and $\gamma \in G(L)_f$, then (c.f. §1.5)

$$\Lambda(\delta(pg\gamma)) = \Lambda(\delta(p)\delta(g)) = \Lambda(\delta(g)).$$

So $\psi(\Lambda) = \Lambda \circ \delta$ is defined on $P_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_f$. Moreover, δ was constructed so that for any $g_\infty \in G_\infty$ we have $jac(g_\infty, o) = jac(\delta(g_\infty), o)$, thus for any $g \in G_{\mathbf{A}}$, we have

$$\Lambda(\delta(g_\infty)) = jac(\delta(g_\infty), o)^k = jac(g_\infty, o)^k.$$

So $\Lambda \circ \delta \in \mathcal{E}_k(G)$.

It is easy to see that the restriction map $\Phi \mapsto \Phi|D_{\mathbf{A}}$ induces a vector space homomorphism of $\mathcal{E}_k(G)$ to $\mathcal{E}_k(D)$ and that for any $\Lambda \in \mathcal{E}_k(D)$ we have $(\Lambda \circ \delta)\,|D_{\mathbf{A}} = \Lambda$, which implies that ψ is an injection. Therefore, to prove the proposition it will suffice to show that the dimension of $\mathcal{E}_k(G)$ is at most as large as that of $\mathcal{E}_k(D)$ and hence it will suffice to show that the set B_D has at least as many elements as B_G.

Let $B_P = P_{\mathbf{Q}}\backslash P_{\mathbf{A}}/P(L)_f P_\infty$. Since P is the semidirect product of D and its unipotent radical U, and since U has the strong approximation property, it follows that a complete set of representatives for the double cosets B_D is also a complete set of representatives for B_P. Since $G_{\mathbf{A}}$ admits an Iwasawa decomposition, $G_{\mathbf{A}} = P_{\mathbf{A}} G(L)_f G_\infty$, and we have

$$G_{\mathbf{A}} = \left(\bigcup_{b \in B_D} P_{\mathbf{Q}} b P(L)_f P_\infty\right) G(L)_f G_\infty = \bigcup_{b \in B_D} P_{\mathbf{Q}} b G(L)_f G_\infty.$$

So the cardinality of B_G is at most that of B_D, and hence is equal to that of B_D. **Q.E.D.**

CHAPTER 2

ADELIC EISENSTEIN SERIES

In this section, we associate Eisenstein series on the adele group $G_{\mathbf{A}}$ to certain Hecke characters of K, and we provide the classical interpretation for these adelic Eisenstein series. We also obtain a useful integral representation of the Eisenstein series by following an approach used successfully by Siegel, Baily, Tsao, and Karel in the case of tube domains.

2.1. Adelic modular forms

An adelic modular form of weight k is a continuous function, F, on the double coset space $G_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_f$ such that for all $g_f \in G_f$ fixed and for $o = (\sqrt{-D}/2, 0, 1)$, the function

$$F(\xi, g_f) = jac(g_\infty, o)^{-k} F(g_\infty g_f)$$

depends only on the point $\xi = g_\infty o$ of $D(R)$ and induces a holomorphic function of ξ. We denote the space of adelic modular forms of weight k for L by $A_k(L)$. The center $Z_{\mathbf{A}}$ acts on this space by taking $F(g)$ to $F(\zeta g)$, and the eigenfunctions of this representation transform via ideal class characters χ of K. Let $A_k(L,\chi)$ denote the eigenspace associated to χ.

It is not hard to see that $A_k(L)$ is isomorphic to the direct product of the vector spaces of classical modular forms of weight k for a complete set of representatives of the classes in the genus of L. Indeed, for an arithmetic subgroup Γ of $G_{\mathbf{Q}}$, let $A_k(\Gamma)$ denote the space of holomorphic functions, f, on $D(R)$ such that $f(\gamma\xi) = jac(\gamma,\xi)^{-k} f(\xi)$ for all $\gamma \in \Gamma$ and $\xi \in D(R)$. This is the space of classical modular forms of weight k with respect to Γ. The proof of the following lemma is standard.

Lemma 2.1. *Let $\{\alpha_i : i = 1, \ldots, h'\} \subset G_f$ be a complete set of representatives for the double cosets, $G_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_f G_\infty$, and let $F \in A_k(L)$. For each i let $\Gamma_i = G(\alpha_i L)$ and let F_i denote the function on $\mathcal{D}(R)$ defined by $F_i(\xi) = F(\xi, \alpha_i)$ then the map*

$$A_k(L) \to \bigoplus_i A_k(\Gamma_i) \qquad\qquad F \mapsto (F_1, \ldots, F_{h'})$$

is an isomorphism.

Proof: First we verify that if $F \in A_k(L)$, then $F_i \in A_k(\Gamma_i)$. Since F_i is a holomorphic on $\mathcal{D}(R)$ by assumption, it will suffice to show that for all $\xi \in \mathcal{D}(R)$ and all $\gamma \in G(\alpha_i L)$, we have

$$F_i(\gamma\xi) = jac(\gamma,\xi)^{-k} F_i(\xi).$$

Let $\xi = g_\infty o$ for some $g_\infty \in G_\infty$ Using the definition of F_i and the invariance of F under left translation by $\gamma \in G_{\mathbf{Q}}$, we find

$$F_i(\gamma\xi) = F(\gamma\xi, \alpha_i) = jac(\gamma g_\infty, o)^{-k} F(\gamma_\infty g_\infty \alpha_i) = jac(\gamma g_\infty, o)^{-k} F(\gamma_f^{-1} g_\infty \alpha_i).$$

Next, let $\lambda = \alpha_i^{-1}\gamma_f\alpha_i$ which by assumption is in $G(L)_f$. Using the right translation invariance of F by $\lambda \in G(L)_f$, we find

$$F(\gamma\xi, \alpha_i) == jac(\gamma g_\infty, o)^{-k} F(g_\infty \alpha_i \alpha_i^{-1} \gamma_f \alpha_i) = jac(\gamma g_\infty, o)^{-k} F(g_\infty \alpha_i).$$

Finally, the cocycle property of jac completes the verification

$$F(\gamma\xi, \alpha_i) == jac(\gamma, g_\infty o)^{-k} jac(g_\infty, o)^{-k} F(g_\infty \alpha_i)$$
$$= jac(\gamma, \xi)^{-k} F(\xi, \alpha_i) = jac(\gamma, \xi)^{-k} F_i(\xi),$$

which shows that $F_i \in A_k(G(\alpha_i L))$.

We have seen that the map $F \mapsto (F_1, \ldots, F_{h'})$ takes $A_k(L)$ to $\bigoplus_i A_k(\Gamma_i)$. Injectivity of this map follows easily from the observation that

$$G_{\mathbf{A}} = \bigcup_{i=1}^{h'} G_{\mathbf{Q}} G_\infty \alpha_i G(L)_f$$

which implies that F_i is identically zero if and only if F restricted to $G_{\mathbf{Q}} G_\infty \alpha_i G(L)_f$ is identically zero.

Surjectivity of the map in question can be verified by constructing an inverse map. Indeed, for $i = 1, \ldots, h'$, let $\phi_i \in A_k(\Gamma_i)$. We will define an element $F \in A_k(L)$ such that $F_i = \phi_i$ for all i. Let $g \in G_{\mathbf{A}}$. By assumption, g can be written as $g_\infty \gamma_f \alpha_i \lambda$, for some $\gamma \in G_{\mathbf{Q}}$, some $\lambda \in G(L)_f$, and some i. For this g, define

$$F(g) = jac(g_\infty, o)^k jac(\gamma_\infty^k, g_\infty o) \phi_i(\gamma_\infty^{-1} g_\infty o).$$

It is straightforward to verify that F depends only on $G_{\mathbf{Q}} g G(L)_f$ of g and is continuous on $G_{\mathbf{A}}$. Moreover, by construction, $jac(g_\infty, o)^{-k} F(g)$ depends, for fixed g_f, only on $\xi = g_\infty o$, and defines a holomorphic function on $\mathcal{D}(R)$. This shows that $F \in A_k(L)$. By construction we also have $F_i = \phi_i$ for all i. **Q.E.D.**

2.2. Adelic Eisenstein series

Adelic Eisenstein series are defined as follows. Let $\mathcal{E}_k(G)$ be, as in §1.7, the vector space of all complex valued functions Φ on the double coset space $P_{\mathbf{Q}} \backslash G_{\mathbf{A}} / G(L)_f$ such that for all $g = g_\infty g_f$ in the adele group, we have $\Phi(g) = jac(g_\infty, o)^k \Phi(g_f)$. For any such Φ, the Eisenstein series associated to Φ is defined by

$$E_\Phi(g) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \Phi(\gamma g).$$

This series is a linear combination of classical Eisenstein series and so is normally convergent for $k > 1$. Indeed,

Lemma2.2. *Let $\{\alpha_i\}$ and Γ_i be as in Lemma 2.1, and let $\Phi \in \mathcal{E}_k(G)$. Then for each i there exist decompositions:*

$$G_{\mathbf{Q}} = \bigcup_{j=1}^{r_i} P_{\mathbf{Q}} \gamma_{ij} \Gamma_i,$$

and, for each i,

$$E_\Phi(\xi, \alpha_i) = \sum_{j=1}^{r_i} \Phi((\gamma_{ij})_f \alpha_i) \, jac(\gamma_{ij}, \xi)^k \, E_{k,\gamma_{ij}}(\xi, \Gamma_i)$$

where for any $g \in G_{\mathbf{Q}}$, and for any subgroup Γ of $G_{\mathbf{Q}}$ commensureable with $G(L)$, $E_{k,g}(\xi, \Gamma)$ is the classical Eisenstein series in $A_k(\Gamma)$ associated to the cusp $P_{\mathbf{Q}} g \Gamma$. This Eisenstein series is defined to be a translation of the Eisenstein series E_k

$$E_{k,g}(\xi, \Gamma) = jac(g, \xi)^{-k} \, E_k(g\xi, {}^g\Gamma) \qquad \text{with } {}^g\Gamma = g \Gamma g^{-1}$$

where $E_k \in A_k(\Gamma)$ is the Eisenstein series associated to the cusp at "infinity" and is defined by the following series

$$E_k(\xi, \Gamma) \sum_{\gamma \in (\Gamma \cap P_{\mathbf{Q}}) \backslash \Gamma} jac(\gamma, \xi)^k.$$

Proof: The existence of the decompositions of $G_{\mathbf{Q}}$ is well-known [2], and is an easy consequence of the local Iwasawa decompositions of §1.5.

The proof of the classical expansion of the adelic Eisenstein series is also straightforward. Indeed, let $g \in G_\infty$ and let $\xi = go$. If $\Phi \in \mathcal{E}_k(G)$, then by definition

$$E_\Phi(\xi, \alpha_i) = jac(g, o)^{-k} E_\Phi(g\alpha_i) = jac(g, o)^{-k} \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \Phi(\gamma g \alpha_i).$$

Next, we use the decomposition of $G_{\mathbf{Q}}$ in the statement of the lemma, and we find that the summation over γ may be replaced by a pair of nested summations:

$$E_\Phi(\xi, \alpha_i) = jac(g, o)^{-k} \sum_{j=1}^{r_i} \sum_{\gamma \in P_{\mathbf{Q}} \backslash P_{\mathbf{Q}} \gamma_{ij} \Gamma_i} \Phi(\gamma g \alpha_i).$$

Now, let $\Gamma_{ij} = \gamma_{ij} \Gamma_i \gamma_{ij}^{-1}$ and let $\Gamma_{ij,\infty} = \Gamma_{ij} \cap P_{\mathbf{Q}}$. It is not hard to verify that

$$P_{\mathbf{Q}} \backslash P_{\mathbf{Q}} \gamma_{ij} \Gamma_i = \bigcup_{\delta \in \Gamma_{ij,\infty} \backslash \Gamma_{ij}} P_{\mathbf{Q}} \delta \gamma_{ij}$$

where the union is disjoint. Applying this observation, and the left $P_{\mathbf{Q}}$–invariance of Φ, we find

$$E_\Phi(\xi, \alpha_i) = jac(g, o)^{-k} \sum_{j=1}^{r_i} \sum_{\delta \in \Gamma_{ij,\infty} \backslash \Gamma_{ij}} \Phi(\delta \gamma_{ij} g \alpha_i).$$

Next, applying the definition of Φ and the cocycle property of jac, we find that for any $\gamma \in G_{\mathbf{Q}}$, we have

$$jac(g,o)^{-k}\Phi(\gamma g \alpha_i) = jac(g,o)^{-k} jac(\gamma g, o)^{-k}\Phi(\gamma_f \alpha_i) = jac(\gamma, \xi)^{-k}\Phi(\gamma_f \alpha_i).$$

In particular, if $\gamma = \delta\gamma_{ij}$ for some $\delta \in \Gamma_i$, then

$$\Phi\left((\delta\gamma_{ij})_f\, \alpha_i\right) = \Phi\left((\gamma_{ij})_f \alpha_i \alpha_i^{-1}(\gamma_{ij}^{-1}\delta\gamma_{ij})_f\, \alpha_i\right) = \Phi\left((\gamma_{ij})_f\, \alpha_i\right).$$

The last equality follows because $\alpha_i^{-1}(\gamma_{ij}^{-1}\delta\gamma_{ij})_f\, \alpha_i \in G(L)_f$. Therefore, we obtain

$$\begin{aligned} E_\Phi(\xi, \alpha_i) &= \sum_{j=1}^{r_i} \sum_{\delta \in \Gamma_{ij,\infty}\backslash \Gamma_{ij}} jac(\delta\gamma_{ij}, \xi)^{-k}\Phi\left((\gamma_{ij})_f \alpha_i\right) \\ &= \sum_{j=1}^{r_i} \Phi\left((\gamma_{ij})_f \alpha_i\right) \sum_{\delta \in \Gamma_{ij,\infty}\backslash \Gamma_{ij}} jac(\delta\gamma_{ij}, \xi)^{-k} \\ &= \sum_{j=1}^{r_i} \Phi\left((\gamma_{ij})_f \alpha_i\right) jac(\gamma_{ij}, \xi)^{-k} \sum_{\delta \in \Gamma_{ij,\infty}\backslash \Gamma_{ij}} jac(\delta, \gamma_{ij}\xi)^{-k} \\ &= \sum_{j=1}^{r_i} \Phi\left((\gamma_{ij})_f \alpha_i\right) jac(\gamma_{ij}, \xi)^{-k} E_{k,\gamma_{ij}}(\xi, \Gamma_i). \end{aligned}$$

Q.E.D.

At this point it is appropriate to remark that the Lemmas 2.1 and 2.2 can be made more explicit using the results of Shimura summarized in Lemma 1.7 from which it follows that the set $\{\alpha_i\}$ is in one-to-one correspondence with the ideal class group CL_K of K. Indeed, if $\{a_i : i = 1, \ldots, h\} \subset K_f^*$ is a complete set of ideal class representatives for K and if for each i, we choose $x_i, y_i \in K_f^*$ such that $x_i^3 y_i^{-2}$ is in the same ideal class as a_i. Then we may take $\alpha_i = d(x_i, y_i) \in D_f$ in the notation of §1.2.

2.3. Restriction to $M_{\mathbf{A}}$

It follows easily from Lemma 1.7 that the Eisenstein series constructed from Hecke characters, $\Lambda \in \mathcal{H}_k(D)$, of D:

$$E_\Lambda(g) = \sum_{\gamma \in P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} \Lambda(\delta(\gamma g))$$

map to a basis of the space spanned by all Eisenstein series E_Φ with $\Phi \in \mathcal{E}_k(G)$. This latter space is isomorphic to the space spanned by all classical Eisenstein series under the isomorphism of Lemma 2.1. Therefore, we consider only the adelic Eisenstein series, E_Λ, in the sequel. Since $\mathcal{H}_k(D)$ is zero unless k is divisible by the order k_0 of O_k^*, we also assume henceforth that $k_0|k$.

Let M be the solvable group TU and recall that $D = Z \times T$ and $P = Z \times M$ (c.f. §1.2). In light of the adelic Iwasawa decomposition: $G_{\mathbf{A}} = M_{\mathbf{A}} Z_{\mathbf{A}} \mathbf{K}^0_{\mathbf{A}}$ (§1.5), any F in $A_k(L, \chi)$ is determined by its restriction to $M_{\mathbf{A}}$. Thus, without loss of generality we can further restrict our attention to the functions $E_\Lambda | M_{\mathbf{A}}$. Recall that any element u of the unipotent radical $U_{\mathbf{A}}$ of $M_{\mathbf{A}}$ can be written in the form $[w, t]$ where $w \in K_{\mathbf{A}}$, $t \in \mathbf{Q}_{\mathbf{A}}$, in the notation of §1.2, and that the maximal torus $T_{\mathbf{A}}$ of $M_{\mathbf{A}}$ is isomorphic to $K^*_{\mathbf{A}}$. Using this notation, we will often write an element m of $M_{\mathbf{A}}$ in the form $[w, t]a$ or $a[w, t]$.

2.4. Fourier-Jacobi coefficients and adelic theta coefficients

In this section we introduce two expansions of adelic modular forms: the Fourier-Jacobi expansion and the adelic theta expansion. These two expansions are related in a straightforward manner. The former is the expansion we wish to study, but the latter has a more direct relationship to classical theta functions and is therefore easier to study.

Recall that the center $N_{\mathbf{A}}$ of $U_{\mathbf{A}}$ is isomorphic to $\mathbf{Q}_{\mathbf{A}}$ via the map $t \mapsto [0, t]$. Thus, $\mathbf{Q}^*_{\mathbf{A}}$ acts on $N_{\mathbf{A}}$ by $r \cdot [0, t] = [0, rt]$. Let λ be the unique continuous character of $N_{\mathbf{Q}} \backslash N_{\mathbf{A}} / N(L)_f$ such that $\lambda([0, t_\infty]) = e(t_\infty)$, where $e(z) = exp(2\pi i z)$.

The Fourier-Jacobi expansion of a modular form $F \in A_k(L, \chi)$ is a series

$$F(m) = \sum_{r \in \mathbf{Q}} F'_r(m)$$

where for each r, F'_r is the function on $U_{\mathbf{Q}} \backslash M_{\mathbf{A}} / M(L)_f$ given by

$$F'_r(m) = \int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} F(nm) \lambda(-r \cdot n) dn$$

where dn is Haar measure on N normalized so that its nonarchimedean component dn_f assigns measure 1 to the open compact subgroup $N(L)_f$, and such that dn_∞ is given by the usual Haar measure dt on $\mathbf{R}$ under the identification $t \mapsto [0, t]$ of $\mathbf{R}$ with N_∞.

The relationship between Fourier-Jacobi coefficients and classical theta functions is rather complex. To describe this relationship, we must first partition $M_{\mathbf{A}}$ into countably many open subsets $M_{\mathbf{A}}(c)$,

$$M_{\mathbf{A}}(c) = \{m = ua : u \in U_{\mathbf{A}}, a \in T_{\mathbf{A}}, (a_f \overline{a}_f) = (c)\},$$

indexed by rational numbers c which are norms of ideals of K. Now, let $F'_{r,c}$ denote the restriction of F'_r to $M_{\mathbf{A}}(c)$. In Chapter 3 we will see that $F'_{r,c}$ is non-zero only if rc is a nonnegative integer and that $F'_{r,c}$ can be identified with a vector of classical theta functions of weight proportional to rc. Thus, for each rational number r, the Fourier-Jacobi coefficient F'_r is associated to an infinite collection of vectors of classical theta functions of varying weights.

Rather than study these somewhat complicated Fourier-Jacobi coefficients directly (which was Shintani's approach), we will investigate a related expansion of F into a series of functions whose relationship with classical theta functions is more direct. For each nonnegative integer ν, let F_ν be the complex valued function on $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}/M(L)_f$ defined by

$$F_\nu(ua) = \int_{N_{\mathbf{A}}} F(nua)\lambda(-\nu\|a_f\|\cdot n)dn, \quad u \in U_{\mathbf{A}}, a \in T_{\mathbf{A}},$$

where we are identifying $T_{\mathbf{A}}$ with $K^*_{\mathbf{A}}$ and $N_{\mathbf{A}}$ with $\mathbf{Q}_{\mathbf{A}}$. Under these identifications, $\|a_f\|$ denotes the rational number which is the norm of the finite idele a_f, and therefore $\nu\|a_f\|\cdot n$ is the rational adele x defined by

$$x_\infty = \nu\|a_f\|n_\infty \quad \text{and} \quad \forall p\; x_p = \nu\|a_f\|n_p.$$

The relationship between the F_ν and the Fourier-Jacobi coefficients F'_r is fairly simple. Indeed, observe that if $\nu = rc$ then

$$F_\nu|M_{\mathbf{A}}(c) = F'_r|M_{\mathbf{A}}(c).$$

This implies that we obtain an expansion for F in terms of the F_ν:

$$F = \sum_{\nu=0}^{\infty} F_\nu,$$

which converges uniformly and absolutely on compact subsets of $M_{\mathbf{A}}$. (Indeed, when both sides are restricted to $M_{\mathbf{A}}(c)$ for any fixed c, we obtain the restriction of the Fourier-Jacobi expansion to $M_{\mathbf{A}}(c)$, which converges normally.) To discriminate between this expansion and the Fourier-Jacobi expansion, we will call this expansion the *adelic theta expansion* of F and we call F_ν the νth *adelic theta coefficient.*

The adelic theta coefficient F_ν is a continuous function on the double coset space $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}/M(L)_f$ which is transformed under left translation by $n \in N_{\mathbf{A}}$ according to the formula:

$$F_\nu(nua) = \lambda(\nu\|a_f\|\cdot n)F_\nu(ua), \quad \forall u \in U_{\mathbf{A}}, a \in T_{\mathbf{A}}.$$

The definition of the adelic theta coefficient F_ν can also be written as

$$F_\nu(m) = \int_{N_{\mathbf{A}}} F(nm)\lambda(-\nu\beta(m)\cdot n)dn, \quad \forall m \in M_{\mathbf{A}},$$

where $\beta : M_{\mathbf{A}} \to \mathbf{Q}^*$ is the function defined by

$$\beta(ua) = \|a_f\|, \quad \forall u \in U_{\mathbf{A}}, a \in T_{\mathbf{A}}.$$

The factor, $\beta(m)$, is included in the definition of the adelic theta coefficients so that they will be left $M_{\mathbf{Q}}$–invariant, which is not the case for the Fourier-Jacobi coefficients.

Also, notice that F_ν is identically zero unless ν is a rational integer. Indeed, let $n \in N_f$, $u \in U_{\mathbf{A}}$, $a \in T_{\mathbf{A}}$, and observe that if $n_1 = ana^{-1}$ then

$$F_\nu(uan) = F_\nu(n_1ua) = \lambda(\nu\|a_f\|\cdot n_1)F_\nu(ua).$$

Recall that we identify $T_{\mathbf{A}}$ with $K_{\mathbf{A}}^*$ and $N_{\mathbf{A}}$ with $\mathbf{Q}_{\mathbf{A}}$. Using these identifications, we have

$$n_1 = a_f \overline{a_f} n, \text{ and so } F_\nu(uan) = \lambda(\nu \|a_f\| a_f \overline{a_f} n) F_\nu(ua).$$

Since F_ν is right $N(L)_f$–invariant, this last equation implies that if F_ν is not identically zero then for all $n \in N(L)_f$ we must have

$$\lambda(\nu \|a_f\| a_f \overline{a_f} n) = 1.$$

In other words, if y is the rational idele $\nu \|a_f\| a_f \overline{a_f} n$, then for all p, we must have $y_p \in O_p$. Since

$$y_p = \nu \|a_f\| a_p \overline{a_p} n_p,$$

and since $\|a_f\| a_p \overline{a_p} \in O_p^*$ for all p, we must have $\nu \in \mathbf{Z}_p$ for all p. Hence $\nu \in \mathbf{Z}$. We will see in the next Chapter, that F_ν is identically zero unless ν is also non-negative, in addition to being intgral.

2.5. Holomorphicity of the coefficients

The holomorphicity property of an adelic modular form F induces a similar, but more involved property for its adelic theta coefficients. To state this property we first need to introduce some notation.

Let $o = (\sqrt{-D}/2, 0, 1)$ be the base point used in the definition of an adelic modular form, and define the complex valued functions ξ_1, ξ_2 on M_∞ by

$$m_\infty \cdot o = (\xi_1(m_\infty), \xi_2(m_\infty), 1).$$

So if $m_\infty = [w, t]a$ then

$$\xi_1(m) = t + (\|a\| + \|w\|)\sqrt{-D}/2 \qquad \xi_2(m) = w.$$

Lemma 2.5. *Let $F \in A_k(L)$ and let $\{F_\nu\}$ be its adelic theta coefficients. Then for $m = m_\infty m_f \in M_{\mathbf{A}}$ with m_f fixed, the function*

$$\tilde{F}_\nu(m) = jac(m_\infty, o)^{-k}\; e(-\nu \|\delta(m_f)\|\, \xi_1(m_\infty))\; F_\nu(m),$$

depends only on $w_\infty = \xi_2(m_\infty)$ in $\mathbf{C}$, and the function it defines on $\mathbf{C}$ is holomorphic.

Proof: This is a standard result. The main idea is that if $m_f = u_f a_f$ is fixed and if $m_\infty \cdot o = (z_\infty, w_\infty, 1)$ then

$$jac(m_\infty, o)^{-k} F_\nu(m) = \int_0^{\mathcal{N}(a_f)} F((z_\infty + s_\infty, w_\infty, 1),\, m_f) e(-(\nu/\mathcal{N}(a_f)) s_\infty) ds_\infty$$

$$= e((\nu/\mathcal{N}(a_f)) z_\infty) \int_{z_\infty}^{z_\infty + \mathcal{N}(a_f)} F((s_\infty, w_\infty, 1),\, m_f) e(-(\nu/\mathcal{N}(a_f)) s_\infty) ds_\infty.$$

The latter integral clearly depends only on w_∞ and since $F(\xi, m_f)$ is a holomorphic function in ξ, the integral defines a holomorphic function in w_∞. **Q.E.D.**

2.6. Adelic theta functions

The adelic theta coefficient F_ν is a member of the complex vector space, $V_{k,\nu}(L)$, of adelic theta functions which consists of all continuous complex–valued functions θ on $M_{\mathbf{Q}} \backslash M_{\mathbf{A}} / M(L)_f$ such that $\theta(nm) = \lambda(\nu\beta(m) \cdot n)\theta(m)$ for all $n \in N_{\mathbf{A}}$, and such that for $m \in M_{\mathbf{A}}$ with m_f fixed, the function

$$\tilde{\theta}(m) = j_{k,\nu}(m)^{-1}\theta(m),$$

where

$$j_{k,\nu}(m) = jac(m_\infty, o)^k \; e(\nu \|\delta(m_f)\| \, \xi_1(m_\infty))$$

depending only on $\xi_2(m_\infty)$ induces a holomorphic function on $N_\infty \backslash M_\infty / T_\infty \cong K_\infty \cong \mathbf{C}$.

2.7. The Siegel-Baily-Tsao-Karel integral

The first step in the Siegel-Baily-Tsao-Karel technique for determining the adelic theta coefficients of Eisenstein series on a tube domain is to use the Bruhat decomposition for $G_{\mathbf{Q}}$ to obtain an expression for the Eisenstein series as a sum over $U_{\mathbf{Q}}$, and then to use this form of the Eisenstein series to explicitly calculate the adelic theta coefficients, which are shown to admit Euler products of a specific form.

The situation is complicated in our non-tube domain case because the unipotent radical of the parabolic subgroup P is not abelian. This is reflected in the formula that the Siegel-Baily-Tsao-Karel method produces for the adelic theta coefficients. Indeed, the νth adelic theta coefficient is expressed as a sum over the quotient of $U_{\mathbf{Q}}$ by its center $N_{\mathbf{Q}}$. The integrand is a function S_Λ, which we call the Siegel function (associated to Λ). We show in Chapter 5 that the Siegel function is a continuous function on $M_{\mathbf{A}} \times \mathbf{Q}_{\mathbf{A}}$ whose values admit Euler products, as in the tube domain case.

Lemma 2.7. *Let $E_\Lambda(m) = \sum_{\nu=0}^{\infty} E_{\Lambda,\nu}(m)$ be the adelic theta expansion for E_Λ. Then*

a) $E_{\Lambda,0}(m) = \Lambda(\delta(m))$, and

b) if ν is a positive integer and ι is the representative of the non-trivial element in the Weyl group of $G_{\mathbf{Q}}$ (given in Lemma 1.2 above), then

$$E_{\Lambda,\nu}(m) = \sum_{u \in N_{\mathbf{Q}} \backslash U_{\mathbf{Q}}} S_\Lambda(um, \nu\|\delta(m_f)\|),$$

where $S_\Lambda(m, r)$ is the function on $M_{\mathbf{A}} \times \mathbf{Q}_{\mathbf{A}}$ given by

$$S_\Lambda(m, r) = \int_{N_{\mathbf{A}}} \Lambda(\delta(\iota nm)) \; \lambda(-r \cdot n) \; dn,$$

and is called the Siegel function associated to Λ.

Proof: By using the Bruhat decomposition $G_{\mathbf{Q}} = P_{\mathbf{Q}} \cup P_{\mathbf{Q}}\iota U_{\mathbf{Q}}$ and the left-$P_{\mathbf{Q}}$ invariance of $\Lambda \circ \delta$, the Eisenstein series: $E_\Lambda(m) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \Lambda(\delta(\gamma m))$ can be expressed as a sum over $U_{\mathbf{Q}}$:

$$E_\Lambda(m) = \Lambda(\delta(m)) + \sum_{u \in U_{\mathbf{Q}}} \Lambda(\delta(\iota um))$$

and hence the integral defining the νth adelic theta coefficient for E_Λ can be separated into two parts: $E_{\Lambda,\nu}(m) = I'_{\Lambda,\nu}(m) + I''_{\Lambda,\nu}(m)$, where

$$I'_{\Lambda,\nu}(m) = \int_{N_{\mathbf{Q}}\backslash N_{\mathbf{A}}} \Lambda(\delta(nm))\ \lambda(-\nu\beta(m)n)dn$$

where we let $\beta(m)$ denote $\|\delta(m_f)\|$, and

$$I''_{\Lambda,\nu}(m) = \int_{N_{\mathbf{Q}}\backslash N_{\mathbf{A}}} \sum_{u\in U_{\mathbf{Q}}} \Lambda(\delta(\iota unm))\lambda(-\nu\beta(m)n)dn$$

The left-$P_{\mathbf{Q}}$ invariance of $\Lambda\circ\delta$ implies that $\Lambda(\delta(nm)) = \Lambda(\delta(m))$ for any $n\in N_{\mathbf{Q}}$ and $m\in M_{\mathbf{A}}$. Thus, $\nu\mapsto I'_{\Lambda,\nu}(m)$ is the Fourier transform of the constant function $\Lambda(\delta(m))$ on the compact group $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}$, which has measure 1. Therefore,

$$I'_{\Lambda,\nu}(m) = \begin{cases} \Lambda(\delta(m)) & \text{if } \nu = 0 \\ 0 & \nu > 0 \end{cases}$$

By splitting the sum over $U_{\mathbf{Q}}$ into one over $N_{\mathbf{Q}}\backslash U_{\mathbf{Q}} \times N_{\mathbf{Q}}$, and moving the sum over $N_{\mathbf{A}}\backslash U_{\mathbf{Q}}$ out of the integral, we obtain:

$$I''_{\Lambda,\nu}(m) = \sum_{u\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \int_{N_{\mathbf{A}}} \Lambda(\delta(\iota unm))\lambda(-\nu\beta(m)n)dn = \sum_{u\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} S_\Lambda(\dot{u}m, \nu\beta(m))$$

Since $\Lambda(\delta(g_\infty g_f)) = jac(g_\infty, o)^k \Lambda(\delta(g_f))$, the integral defining $S_\Lambda(m,0)$ can be written as a product of an integral over N_∞ and one over N_f, the integral over N_∞ being

$$\int_{N_\infty} jac(\iota nm_\infty, o)^k dn = jac(m_\infty, o)^k \int_{-\infty}^{\infty} (t+z)^{-3k} dt$$

where $z\in\mathbf{C}$, is defined by $m_\infty \cdot o = (z,w)$. This integral takes the value zero, and hence

$$I''_{\Lambda,\nu}(m) = \begin{cases} 0 & \text{if } \nu = 0 \\ \sum_{u\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} S_\Lambda(um, \nu\|\delta(m_f)\|) & \text{if } \nu > 0 \end{cases}$$

and this completes the proof of the lemma. **Q.E.D.**

Corollary. *Let ν be a positive integer, then $S_\Lambda(m,\nu\beta(m))$ is a left-$T_{\mathbf{Q}}$ invariant function of m, and*

$$E_{\Lambda,\nu}(m) = \sum_{m_1\in N_{\mathbf{Q}}T_{\mathbf{Q}}\backslash M_{\mathbf{Q}}} S_\Lambda(m_1 m, \nu\beta(m_1 m))$$

where $\beta(ua) = \|a_f\|$ for $u\in U_{\mathbf{A}}$ and $a\in T_{\mathbf{A}}$.

Proof: Since $M_{\mathbf{Q}}$ is the semidirect product of $U_{\mathbf{Q}}$ and $T_{\mathbf{Q}}$, the inclusion map induces a bijection of $N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}$ onto $N_{\mathbf{Q}}T_{\mathbf{Q}}\backslash M_{\mathbf{Q}}$. Therefore, it will suffice to show that the integrand is a left-$T_{\mathbf{Q}}$ invariant function on $M_{\mathbf{A}}$, which is a simple computation: let $x \in T_{\mathbf{Q}}$, $m \in M_{\mathbf{Q}}$, then

$$S_{\Lambda}(xm, \nu\beta(xm)) = \int_{N_{\mathbf{A}}} \Lambda(\delta(\iota n x m))\lambda(-\nu\beta(xm)n)dn$$

Make the substution $n_1 = n/x\overline{x}$, and applying the observations that $dn_1 = dn$, $\iota x = \overline{x}^{-1}\iota$, to obtain

$$\int_{N_{\mathbf{A}}} \Lambda(\delta(\overline{x}^{-1}\iota n_1 m))\lambda(-\nu\beta(m)\|x_f\|x\overline{x}n_1)dn_1$$

The left-$P_{\mathbf{Q}}$ invariance of $\Lambda\delta$ therefore implies the left-$T_{\mathbf{Q}}$ invariance of $S_{\Lambda}(m, \nu\beta(m))$, which proves the Corollary. **Q.E.D.**

2.8. Comparison with the tube domain case

When the Siegel-Baily-Tsao-Karel technique is applied in the case of a tube domain, the unipotent subgroup $U_{\mathbf{Q}}$ is abelian, and so the integral over $N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}$ in Lemma 5.2 disappears and one needs only obtain a formula for the complex number corresponding to $S_{\Lambda}(m, \nu\beta(m))$. This is done in the tube domain case by exhibiting an Euler product for the term corresponding to $S_{\Lambda}(m, \nu\beta(m))$, which allows it to be expressed in terms of special values of Dirichlet L-series. In our case, $S_{\Lambda}(m, r)$ also admits an Euler product for each fixed $m \in M_{\mathbf{A}}$ and $r \in \mathbf{Q}_{\mathbf{A}}$, but the integral over $N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}$ does not. Thus we must resort to other methods.

Our approach is to express the adelic theta coefficients of the Eisenstein series in terms of a particular basis of the graded ring of theta functions. This basis arises from Shintani's work on the adelic theta coefficients of simultaneous eigenforms of the ring of Hecke operators on modular forms. In his work, he decomposes the graded ring of theta functions into a space of imprimitive theta functions and a space of primitive theta functions, in such a way that these two spaces are orthogonal with respect to a certain inner product. Moreover, he gives a neccessary and sufficient condition for a modular form to be a simultaneous eigenform of the ring of Hecke operators in terms of a relation between the primitive and the imprimitive components of the adelic theta coefficients of the modular form. This result allows us to reduce the problem of determining a formula for the adelic theta coefficients of Eisenstein series to the problem of determining the primitive components of the adelic theta coefficients.

Thus, our approach is to choose a particularly nice basis for the space of primitive theta functions in $V_{k,\nu}(L)$ and to explicitly evaluate the inner product $(E_{\Lambda,\nu}, \theta)$ for every θ in this basis. In Chapter 3 we develop some of the fundamental properties of the adelic theta functions and provide a classical interpretation for them. In Chapter 4 we present Shintani's results specialized to our case of interest. In Chapter 5 we explicitly evaluate the function $S_{\Lambda}(m, r)$ and show that for each fixed m it admits an Euler product. In Chapter 6, we use local techniques to show that, if θ is one of the particular basis elements we consider, then

$$(E_{\Lambda,\nu}, \theta) = c_{\Lambda,\theta} l_{\Lambda}(\theta)$$

where $c_{\Lambda,\theta}$ is a monomial of special values of Dirichlet and Hecke L-series, and $l_{\Lambda}(\theta)$ is a simple linear functional on the space of adelic theta functions of level ν, defined in terms of the values of $\tilde{\theta}$ (cf. §2.6) at points in the class group of K (cf. §6.1.2). In Chapter 7, we show that E_Λ is a simultaneous eigenform of the ring of Hecke operators and calculate its eigenvalues. This allows us to apply Shintani's results and obtain a complete formula for the adelic theta coefficients of Eisenstein series. In Chapter 8, we show that our formula can be used to deduce the arithmeticity, in the sense of Shimura, of the Eisenstein series. This is done by using results of Siegel and of Damerell to evaluate the coefficients $c_{\Lambda,\theta}$ and by investigating the arithmeticity of Shintani's decomposition of the space of adelic theta functions.

CHAPTER 3

THETA FUNCTIONS

In this Chapter, we show that the graded ring of adelic theta functions of weight k is isomorphic to a graded ring of theta functions on $\mathbf{C} \times CL_K$ where CL_K is the ideal class group of K and we construct the isomorphism using a Hecke character of weight k. This isomorphism is used in Chapter 8 to deduce the arithmeticity of the Eisenstein series by reducing to the classical case, but the proof of the main theorem does not use this isomorphism.

This chapter is divided into three sections. In the first section we show the the space $V_{k,\nu}(L)$ of adelic theta functions is isomorphic to a space $\mathcal{V}_\nu$ of theta functions defined on $\mathbf{C} \times CL_K$, where CL_K is the ideal class group of K. In the second section we review the theory of classical theta functions and describe the space $\mathcal{V}_\nu$ in the classical terminology. In particular, we describe the graded ring $V_*(H, \psi, A)$ of theta functions associated to a lattice A, a Riemann form H on A, and a second degree character ψ on A with respect to H. In addition we show how to construct a vector space basis for these graded rings using Riemann's theta function. In the third and last section, we define the Siegel inner product on adelic theta functions and express it in terms of the Siegel inner product on classical theta functions.

3.1. Adelic theta functions

The graded ring of adelic theta functions is the graded ring of all those automorphic forms on $M_{\mathbf{A}}$, with respect to a certain factor of automorphy $j_{k,\nu}$, which transform like a adelic theta coefficient under left translation by $N_{\mathbf{A}}$. In this section we recall the definition of adelic theta functions and work out some of the consequences of this definition.

3.1.1. The definition and basic properties of adelic theta functions

Let us first recall the definition of the factor of automorphy $j_{k,\nu}$ (§2.6). Let ξ_1, ξ_2 be the complex valued functions on M_∞ defined by

$$m_\infty \cdot o = (\xi_1(m_\infty), \xi_2(m_\infty), 1),$$

where $o = (\sqrt{-D}/2, 0, 1)$. Thus, if $m_\infty = [w, t]a$ (in the notation of §1.2), then

$$\xi_1(m_\infty) = t + (\|a\| + \|w\|)\sqrt{-D}/2 \qquad \xi_2(m_\infty) = w.$$

The factor of automorphy $j_{k,\nu}$ is then defined for $m \in M_{\mathbf{A}}$ by

$$j_{k,\nu}(m) = jac(m_\infty, o)^k \; e(\nu \|\delta(m_f)\| \, \xi_1(m_\infty)).$$

So if $m = [w,t]a$, we have

$$j_{k,\nu}(m) = (a_\infty \|a_\infty\|)^k \;\; e\left(\nu \|a_f\| \;\left(t_\infty + (\|a_\infty\| + \|w_\infty\|)\sqrt{-D}/2\right)\right).$$

Also let λ be the unique continuous character of $N_{\mathbf{Q}} \backslash N_{\mathbf{A}} / N(L)_f$ such that $\lambda([0,t_\infty]) = e(t_\infty)$. Given this notation we can now define the space of adelic theta functions.

The space $V_{k,\nu}(L)$ of adelic theta functions of weight k and level ν for the lattice L consists of all continuous functions θ on the double coset space $M_{\mathbf{Q}} \backslash M_{\mathbf{A}} / M(L)_f$ which satisfy the following two conditions. First, for any $n \in N_{\mathbf{A}}$, and any $m = ua \in M_{\mathbf{A}}$, we have

$$\theta(nm) = \lambda(\nu \|a_f\| \cdot n)\theta(m),$$

and second, for any $m = m_\infty m_f \in M_{\mathbf{A}}$, with m_f fixed, the function

$$\tilde{\theta}(m) = j_{k,\nu}(m)^{-1}\theta(m),$$

depends only on $w_\infty = \xi_2(m_\infty) \in \mathbf{C}$ and defines there a holomorphic function. We will use the notation

$$\theta(w_\infty, m_f) = \tilde{\theta}([w_\infty, 0]m_f),$$

to denote this holomorphic function. The graded ring $V_{k,*}(L)$ of theta functions of weight k for L is the direct sum of the spaces $V_{k,\nu}(L)$ for all integers ν. In the next lemma we describe the dependence of the holomorphic function $\theta(w_\infty, m_f)$ on the finite adelic element m_f. We will assume that ν and k are rational integers and that k is divisible by k_0, which is the order of the group of units of O_K.

Lemma 3.1.1. *Let $\theta \in V_{k,\nu}(L)$, $w \in \mathbf{C}$, $[v,t] \in U_{\mathbf{Q}}$, $x \in T_{\mathbf{Q}}$, and $\beta \in T_f$. Then*

a) $\theta(w, [v_f, t_f]\beta) = \rho(w, [v_f, t_f]\beta)^{-1}\theta(w - v_\infty, \beta)$ where

$$\rho(w, [v_f, t_f]\beta) = e\left(\nu \|\beta\| t_\infty\right) e\left(\nu \|\beta\| \, \overline{v_\infty} \, (w - v_\infty/2)\tau)\right);$$

b) $\theta(w, x_f\beta) = jac(x_\infty, o)^{-k}\theta(w/x_\infty, \beta)$;

c) θ is determined by the restriction of $\tilde{\theta}$ to the set $\mathbf{C} \times \{\beta_i\}$, where $\{\beta_i\}$ is a complete set of representatives for the ideal classes of K.

Proof: Parts (a) and (b) are elementary computations. For part (c), let $\{\beta_i\}$ be a complete set of representatives for the ideal classes of K, then any $m \in M_f$ can be written in the form $m = [v_f, t_f]x_f\beta_i u$ for some i, where $[v,t] \in U_{\mathbf{Q}}$, $x \in T_{\mathbf{Q}}$, and $u \in U(L)_f$. This follows from the fact that

$$U_f = (U_{\mathbf{Q}} U_\infty \cap U_f)U(\beta L)_f, \qquad T_f = \bigcup_i (T_{\mathbf{Q}} T_\infty \cap T_f)\beta_i T(L)_f.$$

Thus, the function $\theta(w_\infty, m_f)$, for m_f fixed, can be expressed in terms of the function $\theta(w_\infty, \beta_i)$, for some i, by parts (a) and (b). **Q.E.D.**

3.1.2. Theta functions on $\mathbf{C} \times I_K$

Now we show that the graded ring $V_{k,*}(L)$ of adelic theta functions is isomorphic to a certain graded ring of classical theta functions with complex multiplication by K. To construct this isomorphism we need a Hecke character of weight k on the torus T.

Let Λ be a Hecke character on T of weight k (§1.7). So Λ is a continuous homomorphism from $T_\mathbf{Q}\backslash T_\mathbf{A}/T(L)_f$ into $\mathbf{C}^*$ such that $\Lambda(x_\infty) = jac(x_\infty, o)^k$, for all $x_\infty \in T_\infty$. For $\theta \in V_{k,\nu}(L)$, let $\Theta_\Lambda(\theta)$ be the function on $\mathbf{C} \times T_f$ defined by

$$\left(\Theta_\Lambda(\theta)\right)(w, \beta) = \Lambda(\beta)\theta(w_\infty, \beta).$$

The next lemma shows that for $\beta \in T_f$ fixed, the function $\theta_\Lambda(w, \beta)$ is a classical theta function on $\mathbf{C}$ with respect to the ideal (β). The lemma also describes some of the relations among these classical theta functions for various β.

Lemma 3.1.2. *Let $\theta \in V_{k,\nu}(L)$, $\Lambda \in \mathcal{H}_k(T)$, and let $\Theta_\Lambda(\theta)$ be as defined above. For $\beta \in T_f$, let B be the ideal associated to β. Then*

a) For all $b \in B$, $w \in \mathbf{C}$ one has

$$\left(\Theta_\Lambda(\theta)\right)(w + b, \beta) = e\left(-\nu\|\beta_f\|b\bar{b}D/2\right) e\left(-\nu\|\beta_f\|\,\bar{b}\,(w + b/2)\tau)\right) \left(\Theta_\Lambda(\theta)\right)(w, \beta);$$

b) For all $x \in T_\mathbf{Q}$ and $y_f \in T(L)_f$,

$$\left(\Theta_\Lambda(\theta)\right)(w_\infty, x_f\beta y_f) = \left(\Theta_\Lambda(\theta)\right)(w_\infty/x_\infty, \beta),$$

so $\left(\Theta_\Lambda(\theta)\right)(w, \beta)$ defines a function on $\mathcal{M} = \mathbf{C} \times I_K$ where I_K is the group of ideals in K, and this function is invariant under the action of K^* on $\mathcal{M}$ defined by $x \cdot (w, \beta) = (wx_\infty, \beta x_f)$.

Proof: For part (a), observe that if $b \in B$, then $u = [b, b\bar{b}D/2] \in U(\beta L)$, so part (a) follows from part (a) of the previous lemma since $u_f\beta \in \beta U(L)_f$. From part (b) of the previous lemma we also see that for $x \in T_\mathbf{Q}$,

$$\left(\Theta_\Lambda(\theta)\right)(w, x_f\beta) = \Lambda(x_f) jac(x_\infty, o)^{-1}\theta(w, \omega_f\beta).$$

But $\Lambda(x_f) = jac(x_\infty, o)^{-1}$, since $x \in T_\mathbf{Q}$; this proves (b). **Q.E.D.**

This lemma implies that every $\theta \in V_{k,\nu}(L)$ is associated to a function $\Theta_\Lambda(\theta)$ on the space $K^*\backslash\mathcal{M}$. This quotient space is easily seen to be have h components where h is the class number of K. Indeed, if $\{(\beta_i)\}$ is a complete set of representatives for the ideal classes of K, then a fundamental domain for the action of K^* on $\mathcal{M}$ is given by $\mathbf{C} \times \{\beta_i\}$.

3.1.3. Theta functions on $\mathbf{C} \times CL_K$

Motivated by the Lemma 3.1.2, we define certain spaces of theta functions with respect to ideals of K as follows. For any ideal β of K, let $\mathcal{V}_\nu(\beta)$ be the complex vector space of all holomorphic functions g on $\mathbf{C}$ such that for any $b \in (\beta)$ we have

$$g(w + b) = e\left(-\nu\|\beta_f\|b\bar{b}D/2\right) e\left(-\nu\|\beta_f\|\,\bar{b}(w + b/2)\tau\right) g(w),$$

and let $\mathcal{V}_\nu$ be the complex vector space of functions, g, on $\mathcal{M} = \mathbf{C} \times I_K$ which are invariant under the action of K^* and such that for every $\beta \in I_K$ the function $g(w, \beta)$ is in $\mathcal{V}_\nu(\beta)$ and such that for every $x \in K^*$ one has

$$g(w, \beta x_f) = g(w/x_\infty, \beta).$$

Observe that if $g \in \mathcal{V}_\nu$, then for all $\beta \in I_K$ and $\omega \in O_K^*$

$$g(\omega w, \beta) = \omega^k g(w, \beta\omega) = g(w, \beta),$$

since we are assuming that k is divisible by the order, k_0, of O_K^*. We let $\mathcal{V}_\nu{}'(\beta)$ denote the subspace of all $g \in \mathcal{V}_\nu(\beta)$ that are invariant under translation by elements of O_K^*; that is $g(\omega w) = g(w)$ for all $\omega \in O_K^*$.

Lemma 3.1.3. *Let Λ be a Hecke character of weight k on T, and let $\Theta_\Lambda : V_{k,\nu}(L) \to \mathcal{V}_\nu$ be the map defined by*

$$\Theta_\Lambda(\theta)(w, \beta) = \Lambda(\beta)^{-1}\tilde{\theta}([w_\infty, 0]\beta).$$

Then Θ is an isomorphism.

Proof: We have seen that any $\theta \in V_{k,\nu}(L)$ is uniquely determined by the function $\Theta_\Lambda(\theta)$, so Θ_Λ is an injection. We must show it is surjective. This follows from the observation that the connected components of $M_{\mathbf{Q}} \backslash M_{\mathbf{A}}/M(L)_f$ are of the form

$$M_\infty U_{\mathbf{Q}} T_{\mathbf{Q}} \beta T(L)_f U(L)_f,$$

where β ranges over a complete set of representatives of the ideal classes of K. So one must show that the subspace of adelic modular forms supported on one of these connected components maps surjectively onto one of the spaces $\mathcal{V}_\nu{}'(\beta)$, and this is a straightforward verification. **Q.E.D.**

3.2. Classical theta functions

In this section, we provide a brief review of the theory of classical theta functions on $\mathbf{C}$ which allows us to calculate the dimensions of the spaces $\mathcal{V}_\nu(\beta)$ and in principle allows one to calculate the dimension of $V_{k,\nu}(L)$ although we do not carry that calculation through at this point.

3.2.1. Riemann forms, second degree characters, and classical theta functions

Classical theta functions are classified by three parameters: a lattice in a complex vector space, a Riemann form on that vector space with respect to the lattice, and a second degree character on the lattice with respect to the Riemann form. We consider here only the case of theta functions on $\mathbf{C}$.

A Riemann form for a lattice A in $\mathbf{C}$ is a positive definite Hermitian form H on $\mathbf{C}$ such that, for all ξ_1, ξ_2 in A, the imaginary part of $H(\xi_1, \xi_2)$ is a rational integer. A second degree character, ψ, on A with respect to a Riemann form H is a function from A into $\mathbf{C}^*$ such that for all $\xi_1, \xi_2 \in A$ one has

$$\psi(\xi_1 + \xi_2) = \psi(\xi_1)\psi(\xi_2)e(Im(H(\xi_1, \xi_2))/2).$$

(Note that for any ξ_1, $\xi_2 \in A$, that $e(Im(H(\xi_1,\xi_2))/2) = \pm 1$, which implies that ψ^2 is a character of A.) From the data (H, A, ψ) we can construct a factor of automorphy

$$u_\xi(w) = e((1/2i)\, H(\xi, w + \xi/2))\psi(\xi),$$

which satisfies the cocycle relation:

$$u_{\xi_1+\xi_2}(w) = u_{\xi_1}(w + \xi_2)u_{\xi_2}(w),$$

for all ξ_1, ξ_2 in A. Indeed, the peculiar definition of the second degree character is precisely what is needed to make u_ξ a factor of automorphy.

The space $V(H, \psi, A)$ of theta functions with respect to this factor of automorphy is the complex vector space of all holomorphic functions, g, on $\mathbf{C}$ such that

$$g(w + \xi) = u_\xi(w)g(w),$$

for all $\xi \in A$. The graded ring of theta functions of type (H, ψ, A) is defined by

$$V_*(H, \psi, A) = \bigoplus_n V(nH, \psi^n, A).$$

We will recall how to calculate the dimension of this vector space $V(nH, \psi^n, A)$.

Observe that if H is a Riemann form for A, ψ is a second degree character on A for H, and ν is any positive integer, then νH is a Riemann form for A and ψ^ν is a second degree character on A with respect to νH. Since we are only considering one dimensional theta functions, for each lattice A there is a minimal Riemann form H_A such that every other Riemann form for A is an integer multiple of H_A. In the next lemma we write down this minimal Riemann form.

Lemma 3.2.1. *Let A be a lattice in $\mathbf{C}$ and let (ω_1, ω_2) be a basis of A such that the complex number $z = \omega_1/\omega_2$ has positive imaginary part. Then, the minimal Riemann form for A is*

$$H_A(x, y) = \frac{\overline{x}y}{Im(\omega_1\overline{\omega}_2)} = \frac{\overline{x}y}{Im(z)|\omega_2|^2}.$$

Moreover, all second degree characters on A with respect to the Riemann form νH_A have the form

$$\psi(a\omega_1 + b\omega_2) = e(\nu ab/2 + ar + bs),$$

for real numbers r, s. In particular, if A is an ideal in the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$, then the minimal Riemann form on A is given by

$$H_A(x, y) = \overline{x}y\frac{2}{\sqrt{D}\mathcal{N}(A)},$$

where $\mathcal{N}(A)$ is the norm of the ideal A, and all second degree characters on A with respect to nH_A are of the form

$$\psi(\xi) = \chi(b)\psi_A(b)^n \quad \text{for} \quad \psi_A(\xi) = e(b\overline{b}/2),$$

where χ is a character of A.

Proof: This is an elementary calculation. Indeed, let $z = \omega_1/\omega_2$ and observe that if $\xi_1, \xi_2 \in A$ and $\xi_1/\omega_2 = az+b$ and $\xi_2/\omega_2 = cz+d$, then

$$Im(\overline{\xi}_1\xi_2)/|\omega_2|^2 = -Im(z)(ad-bc),$$

from which the first part of the lemma follows immediately. As to the second part, if A is an ideal of K, and $H(x,y) = \overline{x}yh$ is a Riemann form, then

$$Im(H(\xi_1,\xi_2)) = \mathcal{T}(\xi_1\overline{(\xi_2\sqrt{-D})})\frac{h}{2\sqrt{D}},$$

where $\mathcal{T}(x) = x + \overline{x}$ is the trace map. Thus, if $\mathcal{D}_K$ denotes the different of K over $\mathbf{Q}$, then $\mathcal{D}_K = \tau O_K$ and so $\mathcal{N}(\mathcal{D}) = D$ and $\mathcal{T}(\mathcal{D}_K^{-1}) = \mathbf{Z}$, so

$$Im(H(A,A)) = \mathcal{N}(A)\,Im(H(O_K,O_K)) = \mathcal{N}(A)\mathcal{T}(\tau O_K)h/2\sqrt{D}$$

$$= \mathcal{N}(A)D\mathcal{T}(\tau^{-1}O_K)h/2\sqrt{D} = \mathcal{N}(A)D\mathbf{Z}h/2\sqrt{D},$$

so we must have $\mathcal{N}(A)Dh/(2\sqrt{D}) \in \mathbf{Z}$ as claimed. The assertions concerning second degree characters are equally easy. **Q.E.D.**

Corollary. *Notation as above, $\mathcal{V}_\nu(A) \cong V(\nu DH_A, \psi_A^{\nu D}, A)$.*

3.2.2. An explicit basis for the space of classical theta functions

We will now show how the Riemann theta function can be used to construct a basis of the space $V(H,\psi,A)$.

Lemma. 3.2.2. *Let θ be Riemann's theta function which is defined by*

$$\theta(w,z;r,s) = \sum_{n=-\infty}^{\infty} e((n+r)^2z/2 + (n+r)(w+s)) \qquad w,z \in \mathbf{C}, Im(z) > 0, r,s \in \mathbf{R},$$

let ϕ be the related function defined by

$$\phi(w,z;r,s) = e(w^2/(4iIm(z)))\theta(w,z;r,s),$$

and let A be a lattice with basis $\Omega = (\omega_1,\omega_2)$ such that $Im(\omega_1/\omega_2) > 0$. Then for any integer n, and any real numbers r,s, the n holomorphic functions

$$g_j(w) = \phi(nw/\omega_2, n\omega_1/\omega_2; r + j/n, -ns) \qquad j \in \mathbf{Z}/n\mathbf{Z},$$

form a basis of the space $V(nH_A, \psi^n_{\Omega,r,s}, A)$ where the second degree character is given by

$$\psi_{\Omega,r,s}(a\omega_1 + b\omega_2) = e(ab/2 + as + br).$$

Proof: This is a classical result. See for example [5, Ch. 2, p.75] or [11, §1]. **Q.E.D.**

This lemma implies that $dim_{\mathbf{C}}(\mathcal{V}_\nu(\beta)) = \nu D$, and so the dimension of $\mathcal{V}_\nu$ is at most $h\nu D$ where h is the class number of K. In principle we could calculate the dimension of $\mathcal{V}_\nu$ exactly since it is h times the dimension of $\mathcal{V}_\nu{}'(\beta)$ and this latter space is the subspace of even theta functions in $\mathcal{V}_\nu(\beta)$.

3.3. The Siegel inner product

In this section we define the Siegel inner product on classical and adelic theta functions and show that the map Θ_Λ between the spaces of adelic and classical theta functions is an isometry, up to an explicit constant.

3.3.1. The Siegel inner product on classical theta functions

The Siegel inner product on the space $V(H, \psi, A)$ of classical theta function is defined by

$$(g, g') = \int_{\mathbf{C}/A} \overline{g(w)} g'(w) e^{-\pi H(w,w)} dw/m(A),$$

where $dw = dxdy$ is the standard Haar measure on $\mathbf{C}$ under the identification with $\mathbf{R}^2$ by $w = x + iy$, and $m(A)$ is the measure of a fundamental domain for A. It is easy to see that the integrand is a continuous, doubly periodic function on $\mathbf{C}$ with period lattice A and so the integral is well defined. This inner product was used by Siegel in a proof of the transformation formula for the action of the symplectic group on theta functions [16],[5,p. 79].

Lemma 3.3.1. *Let $\{g_i\}$ be the basis of $V(nH_A, \psi^n, A)$ constructed in Lemma 3.2.2, where A is a lattice with basis (ω_1, ω_2), $z = \omega_1/\omega_2$ has positive imaginary part, and H_A is the minimal Riemann form for A. Then these basis elements are orthogonal with respect to the Siegel inner product and their norms are given by $(g_i, g_i) = 1/\sqrt{2ny}$, where y is the imaginary part of z.*

Proof: This can be proved by direct calculation. See for example, [5, p. 80]. **Q.E.D.**

Thus, the Siegel inner product on the space $\mathcal{V}_\nu(\beta)$ of classical theta functions defined in §3.1.3, is given by the integral

$$(g, g')_\beta = \int_{\mathbf{C}/(\beta)} \overline{g(w)} g'(w)\ e^{-\pi\nu D\, H_\beta(w,w)} dw\, \frac{2}{(\mathcal{N}(\beta)\sqrt{D})}, \qquad g, g' \in \mathcal{V}_\nu(\beta),$$

where we have used the fact that the measure of a fundamental domain for β is $\mathcal{N}(\beta)\sqrt{D}/2$.

We can extend this inner product to the space $\mathcal{V}_\nu$ by choosing a complete set of representatives β_i for the ideal classes of K and defining, for $g, g' \in \mathcal{V}_\nu$:

$$(g, g') = \sum_{i=1}^{h} (g_{\beta_i}, g'_{\beta_i})_{\beta_i},$$

where g_β is by definition the element of $\mathcal{V}_\nu(\beta)$ given by $g_\beta(w) = g(w, \beta)$. In fact, this inner product is independent of the choice of representatives β_i, as can easily be shown by verifying that for any $\beta \in I_K$ and $x \in T_{\mathbf{Q}}$ we have

$$(g_{x\beta}, g'_{x\beta})_{x\beta} = (g_\beta, g'_\beta)_\beta.$$

Thus, we can obtain an inner product on $V_{k,\nu}(L)$ by pulling back this inner product via the map Θ_Λ. This pulled-back inner product is independent of the choice of Λ. Indeed, any other Hecke character Λ' on T of weight k, differs from Λ by a Hecke character of weight 0.

3.3.2. The Siegel inner product on adelic theta functions

We can define an inner product on $V_{k,\nu}(L)$ more directly by simply integrating over the domain $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}$ with respect to an appropriately normalized right Haar measure dm:

$$(\theta_1, \theta_2) = \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \overline{\theta_1(m)}\theta_2(m)dm.$$

To make this precise we will explicitly normalize the right Haar measure dm. Since $M_{\mathbf{A}}$ is a semidirect product, its right Haar measure can be defined in terms of a multiple integral:

$$\int_{M_{\mathbf{A}}} f(m)dm = \int_{K^*_{\mathbf{A}}}\int_{K_{\mathbf{A}}}\int_{\mathbf{Q}_{\mathbf{A}}} f([w,t]a)dtdwda,$$

where dt,dw,da are Haar measures on $\mathbf{Q}_{\mathbf{A}}$,$K_{\mathbf{A}}$, and $K^*_{\mathbf{A}}$ respectively. Furthermore, the nonarchimedean components of these Haar measures are normalized so that they assign measure 1 to the open compact subgroups $\mathbf{Z}_f$, $\vee_f$, and O^*_f, respectively. In addition, the Haar measures on the archimedean components $\mathbf{R}$,$\mathbf{C}$, and $\mathbf{C}^*$ are normalized as follows: dt_∞ is the usual Haar measure on $\mathbf{R}$,

$$\int_{\mathbf{C}} f(w_\infty)dw_\infty = \int_{\mathbf{R}}\int_{\mathbf{R}} f(x+iy)dxdy,$$

$$\int_{\mathbf{C}^*} f(a_\infty)da_\infty = \int_0^1\int_0^\infty f(re^{2\pi is})r^{-1}drds,$$

where dx,dy,dr, and ds are the usual Haar measures on $\mathbf{R}$. Thus the maximal compact subgroup of $\mathbf{C}^*$ has measure 1 and the quotient of $\mathbf{C}$ by an ideal β in K has measure $\mathcal{N}(\beta)\sqrt{D}/2$.

We extend this inner product to the graded ring $\bigoplus_\nu V_{k,\nu}(L)$ by letting the spaces of theta functions of different levels ν be orthogonal subspaces of the direct sum.

Proposition 3.3.2. *Let $\theta, \theta' \in V_{k,\nu}(L)$, and let $\Theta_\Lambda : V_{k,\nu}(L) \to \mathcal{V}_\nu$ be the isomorphism defined in §3.1. Then*

$$(\theta, \theta') = \frac{\sqrt{D}}{2k_0}\, c_0\left(\Theta_\Lambda(\theta), \Theta_\Lambda(\theta')\right),$$

where k_0 is the order of O^*_K and the constant c_0 has the value:

$$c_0 = \int_{K^*_\infty} \|a\|^{3k} e(\nu\tau\|a\|)da = \frac{1}{2}\,\frac{(3k-1)!}{(2\pi\nu\sqrt{D})^{3k}}.$$

Proof: Since $U_{\mathbf{A}}$ is a closed, normal, unimodular subgroup of the locally compact group $M_{\mathbf{A}}$, and $M_{\mathbf{Q}}$ is a discrete unimodular subgroup of $M_{\mathbf{A}}$, and the modulus of left translation by $m \in M_{\mathbf{Q}}$ is 1, we can decompose the Siegel inner product into a composition of integrals:

$$(\theta, \theta') = \int_{M_{\mathbf{Q}} \backslash M_{\mathbf{A}}} \overline{\theta}\theta'(m)dm = \int_{K^* \backslash K^*_{\mathbf{A}}} \int_{U_{\mathbf{Q}} \backslash U_{\mathbf{A}}} \overline{\theta}\theta'(ua)duda.$$

Since θ and θ' are right-invariant with respect to $M(L)_f$, the integrand is a right-invariant function of u with respect to $U(a_f L)_f$, so the inner integral can be written as

$$meas(U(a_f L)_f) \int_{U_{\mathbf{Q}} \backslash U_{\mathbf{A}} / U(a_f L)_f} \overline{\theta}\theta'(\dot{u}a)d\dot{u} = \|a_f\|^2 \int_{U(a_f L) \backslash U_\infty} \overline{\theta}\theta'(u_\infty a)du_\infty.$$

Since

$$\overline{\theta}\theta'([w_\infty, t_\infty]a) = |e(\nu \|a_f\| t_\infty)|^2 \, \overline{\theta}\theta'([w_\infty, 0]a)$$

is independent of $t_\infty \in N_\infty$, we can first integrate over

$$(N_\infty \cap U(a_f L)) \backslash N_\infty \cong \mathcal{N}(a_f) \backslash \mathbf{Q}_\infty,$$

which has measure $|\mathcal{N}(a_f)| = \|a_f\|^{-1}$, and then integrate over

$$(U(a_f L)/N(a_f L)) \backslash (U_\infty / N_\infty) \cong (a_f) \backslash \mathbf{C}.$$

The previous integral becomes

$$\int_{(a_f) \backslash \mathbf{C}} \overline{\theta}\theta'([w_\infty, 0]a)dw_\infty / |\mathcal{N}(a_f)|.$$

Next we rewrite the integral in terms of the classical theta functions $g = \Theta_\Lambda(\theta)$ and $g' = \Theta_\Lambda(\theta')$:

$$\overline{\theta}\theta'([w_\infty, t_\infty]a) = \overline{g}g'(w_\infty, a_f) \; e(\nu\tau \|a\|) \; e(\nu\tau \|a_f\| |w_\infty|^2) \; \|a_\infty\|^{3k},$$

to obtain an expression for the inner product of adelic theta functions in terms of the Siegel inner products of their classical components:

$$(\theta, \theta') = \int_{K^* \backslash K^*_{\mathbf{A}}} \|a\|^{3k} \, e(\nu\tau \|a\|) \, I_{a_f}(g, g')da^*,$$

where $I_{a_f}(g, g')$ depends only on the ideal class of the finite idele a_f: $I_{a_f}(g, g') =$

$$\|a_f\|^{-3k} \int_{(a_f) \backslash \mathbf{C}} \overline{g}g'(w_\infty, a_f) e(\nu\tau \|a_f\| |w_\infty|^2) dw_\infty / |\mathcal{N}(a_f)| = \frac{\sqrt{D}}{2\|a_f\|^{3k}} (g_{a_f}, g'_{a_f})_{a_f}.$$

Finally, since $I_{a_f}(g,g')$ is constant, as a function of $a \in K^*\backslash K^*_{\mathbf{A}}/O^*_f$ along each connected component, we can decompose the integral into a sum over the ideal class group $K^*\backslash K^*_{\mathbf{A}}/O^*_f K^*_\infty$:

$$(\theta,\theta') = \int_{CL_K} c_{\beta_f} I_{\beta_f}(g,g') d\beta^*_f,$$

where

$$c_{\beta_f} = \frac{\sqrt{D}}{2} \int_{O^*_K\backslash K^*_\infty} \|a_\infty \beta_f\|^{3k} e(\nu\tau\|a_\infty\beta_f\|) da^*_\infty$$

$$= \frac{\sqrt{D}}{2} k_0^{-1} \int_0^{2\pi}\int_0^\infty r^{3k} e^{-2\pi\nu\sqrt{D}r^2} r^{-1} dr ds = \frac{\sqrt{D}}{2} \frac{(3k-1)!}{2k_0(2\pi\nu\sqrt{D})^{3k}}$$

is independent of β_f. Thus $(\theta,\theta') = c_0\,(\Theta(\theta),\Theta(\theta'))$ **Q.E.D.**

CHAPTER 4

SHINTANI'S EIGENFUNCTION THEOREM

In this Chapter we summarize the main results of Shintani [13] in which he characterizes those adelic modular forms which are simultaneous eigenfunctions of the ring of Hecke operators in terms of certain relations among their Fourier-Jacobi coefficients. For completeness, we also include proofs of those results of his which are used in our main theorem. Shintani worked in the more general context of vector-valued modular forms on a model of $GU(2,1)$ defined over a totally real number field. We adapt his proofs to the special cases treated here. All of the results in this section are due to Shintani, but they appear here in less generality using a slightly different notion of theta function. In particular, rather than consider Fourier-Jacobi coefficients, we consider the closely related adelic theta coefficients and adapt the proofs accordingly. (c.f. §2.4 for a discussion of the relationship between these two systems of coefficients for modular forms).

Shintani's main result is a characterization of those modular forms which are simultaneous eigenfunctions of the Hecke operators in terms of a relation on their adelic theta coefficients. To do this he first develops a theory of "newforms" on the graded ring of theta functions. This is done by introducing a canonical injection

$$l(a) : V_{k,\nu}(L) \to V_{k,\nu\mathcal{N}(a)}(L),$$

for any integral ideal a of K, where $\mathcal{N}(a)$ is the norm of the ideal a. The space of oldforms is the space spanned by the images of the injections $l(a)$ for all integral ideals $a \neq O_K$, and the space, $V_{k,\nu}(L)^0$, of primitive theta functions is the orthogonal complement of the subspace of oldforms in $V_{k,\nu}(L)$ with respect to the Siegel inner product. Next, he constructs a representation of the group of norm 1 ideals prime to ν on $V_{k,\nu}(L)$ and uses this representation to decompose the space of primitive theta functions into eigenspaces $V_{k,\nu}(L)_\kappa$ on which the representation acts by scalar multiplication by a Hecke character κ. He further decomposes this space using certain projections of $V_{k,\nu}(L)$ associated to subsets, Σ, of primes which ramify in K, thereby obtaining a decomposition:

$$V_{k,\nu}(L)^0 = \bigoplus_{\kappa,\Sigma} V_{k,\nu}(L)^0_{\kappa,\Sigma}.$$

We let Ξ_ν denote the set of all triples $\xi = (\nu, \kappa^*, \Sigma)$ such that the corresponding eigenspace, $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$, is non-zero, and we let Ξ be the union of the Ξ_ν for all positive integers

ν. This decomposition is then used to prove that there is a direct sum decomposition of the graded ring of adelic theta functions:

$$V_{k,*}(L) = \bigoplus_{\nu=0}^{\infty} V_{k,\nu}(L) = \bigoplus_{\nu=0}^{\infty} \bigoplus_{a \in I+} l(a) \left(V_{k,\nu}(L)^0\right),$$

where a ranges over all integral ideals of K. This decomposition can then be used to construct a system of coordinates for modular forms which associates to each modular form a collection of formal Dirichlet series indexed by a basis of the primitive theta functions. Indeed, if we choose, for each $\xi \in \Xi$, an orthonormal basis $\mathcal{B}_\xi$ for the eigenspace space V_ξ and if we let $\mathcal{B}$ be the union of all of these bases $\mathcal{B}_\xi$ then the set of all translates $\{l(a)\theta\}$ of these basis elements by integral ideals a of K forms a basis of the graded ring of theta functions:

$$\bigoplus_{\nu=0}^{\infty} V_{k,\nu}(L) = \bigoplus_{\xi \in \Xi} \bigoplus_{\theta \in \mathcal{B}_\xi} \bigoplus_{a \in I+} \mathbf{C}\left(l(a)\theta\right).$$

Thus, there is a finite direct sum decomposition of each of the spaces $V_{k,\nu}(L)$ into translates of primitive eigenspaces:

$$V_{k,\nu}(L) = \bigoplus_{\substack{a \in I+ \\ c \in \mathbf{Z} \\ a\bar{a}c = \nu}} \bigoplus_{\xi \in \Xi_c} \bigoplus_{\theta \in \mathcal{B}_\xi} \mathbf{C}\left(l(a)\theta\right).$$

Since the adelic theta coefficients of any modular form F on $G_{\mathbf{A}}$ can be written uniquely as a linear combination of these basis elements, we can write F as a series of the form

$$F|M_{\mathbf{A}} = \sum_{\theta \in \mathcal{B}} \sum_{a \in I+} z_\theta(a) \left(l(a)\theta\right),$$

where a runs over all integral ideals of K and c_θ is a complex valued function on the monoid of integral ideals. This system of coordinates can be written more suggestively as

$$F = \sum_{\theta \in \mathcal{B}} Z_\theta \cdot \theta \qquad Z_\theta = \sum_{a \in I+} z_\theta(a) l(a),$$

where Z_θ is a formal Dirichlet series of operators on the graded ring of theta functions.

Shintani's main theorem asserts that a modular form F on $G_{\mathbf{A}}$ is a simultaneous eigenform F of the Hecke operators with eigenvalues λ if and only if *each* of the formal Dirichlet series Z_θ admits an Euler product of the form

$$Z_\theta = z_\theta(1) \prod_p Z_{\xi,\lambda,p} \qquad \theta \in \mathcal{B}_\xi,$$

where the local factors $Z_{\xi,\lambda,p}$ are explicit rational functions in the Shintani operators $\{l(\pi) : \pi | p\}$. These rational functions are units in the ring of formal power series and are given explicitly in Shintani's main theorem which is stated below (Theorem 4.8). The theorem is proved by showing that the Hecke operators on modular forms induce an action on the formal Dirichlet series. This action is then used to show that the Dirichlet series admit Euler products if and only if they are eigenfunctions with respect to this action.

Shintani's theorem shows that to obtain a formula for the adelic theta coefficients of modular forms which are simultaneous eigenfunctions of the Hecke operators it is neccessary and sufficient to determine the eigenvalues and the primitive components. In Chapters 5 and 6, we develop an explicit formula for the primitive components of the adelic coefficients of the Eisenstein series E_Λ and in Chapter 7 we calculate their eigenvalues, thereby obtaining an explicit formula for the adelic theta coefficients, and hence also an explicit formula for the Fourier-Jacobi coefficients.

The summary of Shintani's results presented in this chapter will be organized as follows: In §4.1 we define the Shintani operators and prove a lemma concerning the composition of certain of these operators. In §4.2, we show how Shintani's operators can be used to construct a representation of the group of ideals prime to ν on the space of theta functions of level ν, and we use this representation to decompose the space of theta functions into eigenspaces of this representation. In §4.3, we introduce the projections associated to subset of the set primes which ramify in K and use them to further decompose these eigenspaces. In §4.4, we show that the images of the spaces of primitive functions under the integral Shintani operators forms a collection of independent subspaces. The action of Hecke operators on the spaces of modular forms $A_k(L,\chi)$ is discussed in §4.5, and the effect of these Hecke operators on the adelic theta coefficients of the modular forms is described in §4.6. In §4.7 we define the formal Dirichlet series associated to a modular form. Shintani's characterization of simultaneous eigenfunctions of the Hecke operators in terms of their adelic theta coefficients is given in the last section.

4.1. Shintani operators and primitive theta functions

In this section we define a collection of operators $l(x)$ on adelic theta functions indexed by fractional ideals x of K and we show how they can be used to define the primitive theta functions. We will call these operators "Shintani operators" to avoid any confusion with the Hecke operators on modular forms. These operators will be used in §4.2 to define a representation of the group of ideals prime to ν on $V_{k,\nu}(L)$.

For an idele $x \in K^*_{\mathbf{A}}$, let $l(x): V_{k,\nu}(L) \to V_{k,\nu\mathcal{N}(x)}(L)$ be the Shintani operator defined by

$$(l(x)\theta)(m) = \int_{U(L)_f} \theta(mux^{-1})du,$$

where $\mathcal{N}(x)$ is the norm of the ideal associated to x.

Since $l(x)$ is the identity map for $x \in O_f^*$, $l(x)$ depends only on the ideal (x) associated to x and so l induces a map (which is not a homomorphism) from the group I_K of fractional ideals of K to the ring of endomorphisms of the graded ring of theta functions:

$$l : I_K \to End_{\mathbf{C}}(\bigoplus_\nu V_{k,\nu}(L)).$$

If $a \in I_K$ is an integral ideal then it is easy to see that $l(a)$ is an injection, since $(l(a)\theta)(m) = \theta(ma)$, and that $l(a^{-1}) \circ l(a)$ is the identity map on the graded ring of theta functions $V_{k,*}(L)$.

We define the subspace of oldforms in $V_{k,\nu}(L)$ to be the subspace spanned by the images $l(a)(V_{k,\nu/\mathcal{N}(a)}(L))$ for the finite set of proper integral ideals a such that $\mathcal{N}(a)|\nu$. Analogously to the case of modular forms, we can define the subspace of primitive theta functions in $V_{k,\nu}(L)$ to be the orthogonal complement of the space of oldforms with respect to the Siegel inner product, we will denote the subspace of primitive theta functions by $V_{k,\nu}(L)^0$.

Since $l(a)(l(b)\theta) = l(ab)\theta$ for all integral ideals a, b, and all $\theta \in V_{k,*}(L)$, we see that the images of the primitive theta functions under the integral Shintani operators spans the graded ring of theta functions, that is,

$$V_{k,\nu}(L) = \sum_{\substack{\mathcal{N}(a)c=\nu \\ a\in I+ \\ c\in \mathbf{Z}}} \left(l(a)V_{k,c}(L)^0\right),$$

or more simply

$$V_{k,*}(L) = \sum_{\nu} \sum_{a\in I+} \left(l(a)V_{k,\nu}(L)^0\right),$$

where the inner sum is over the (infinite) set of all integral ideals a of K. We will see later that the sum on the right hand side is actually a direct sum and hence that every theta function can be expressed uniquely as a linear combination of elements of the form $l(a)\theta'$ where θ' is primitive.

The Shintani operators are nicely behaved with respect to the Siegel inner product. Indeed, let $x \in K_f^*$, $\theta \in V_{k,\nu}(L)$ and $\theta' \in V_{k,\nu\mathcal{N}(x)}(L)$, then

$$(l(x)\theta, \theta') = \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \overline{\theta(m)}\theta'(mx)dm = (\theta, l(x^{-1})\theta').$$

It follows from this property that the subspace of primitive theta functions can be defined more intrinsically to be the subspace consisting of all θ such that $l(a^{-1})\theta = 0$ for all integral ideals a such that $\mathcal{N}(a)|\nu$.

As was mentioned before, the map l is not a homomorphism. We conclude this section, by investigating the compositions $l(x)l(y)\theta$ for $\theta \in V_{k,\nu}(L)$ under certain conditions on x and y. The following proposition is the main computational tool in this chapter.

Proposition 4.1. *[13, §2] Let $\theta \in V_{k,\nu}(L)$. We have the following*

a) *Let $x, y \in I_K$ be ideals such that the numerator of x and the denominator of y are relatively prime, and such that $\nu\mathcal{N}(y)$ is an integral ideal; then $l(xy)\theta = l(x)\left(l(y)\theta\right)$.*

b) *Let a be an integral ideal which is relatively prime to ν and to $\overline{a}$, and let $x = a/\overline{a}$; then $l(x)l(x^{-1})\theta = \mathcal{N}(a)^{-1}\theta$.*

Proof: First we prove (a). By definition, the composition $l(x)\left(l(y)\theta\right)(m)$ is given by the double integral

$$\int_{U(L)_f}\int_{U(L)_f} \theta(mu_1x^{-1}u_2y^{-1})du_2du_1 = \int_{U(L)_f}\left(\int_{U(L)_f} \theta(mu_1x^{-1}u_2y^{-1})du_1\right)du_2.$$

To prove (a) it will clearly suffice (since $U(L)_f$ is a group) to show that for every fixed $u_2 \in U(L)_f$ there is a $u_3 \in U(L)_f$ such that

$$\theta(mu_1x^{-1}u_2y^{-1}) = \theta(mu_1u_3x^{-1}y^{-1}).$$

Observe that if $u \in U_f$, $n \in N(L)_f$, $a, y \in T_f$, and if $\nu\mathcal{N}(y)$ is an integral ideal, then

$$\theta(uany^{-1}) = \theta((a\overline{a} \cdot n)uay^{-1}) = \lambda(\nu\|a/y\|a\overline{a} \cdot n)\theta(uay^{-1}),$$

and since $\|a/y\| = \mathcal{N}(y/a)^{-1}$, we have

$$\theta(uany^{-1}) = \lambda\left(\nu\mathcal{N}(y)\frac{(a\overline{a})}{\mathcal{N}(a)}\right) \cdot n)\ \ \theta(uay^{-1}) = \theta(uay^{-1}).$$

Thus, it will suffice to show that for u_2 fixed, there are $u_3, u_4 \in U(L)_f$ and $n \in N(L)_f$ such that

$$x^{-1}u_2y^{-1} = x^{-1}\left(xu_3x^{-1}\right)\left(y^{-1}u_4y\right)ny^{-1}.$$

In other words, we must show that there is an inclusion

$$x^{-1}U(L)_fy^{-1} \subset x^{-1}\left(xU(L)_fx^{-1}\right)\left(y^{-1}U(L)_fy\right)\mathcal{N}(L)_fy^{-1},$$

which can be written more simply as $U(L)_f \subset U(xL)_fU(y^{-1}L)_f\mathcal{N}(L)_f$. Since the numerator of x and the denominator of y are relatively prime, the sum of the ideals (x) and (y^{-1}) contains O_K. Thus, if $u_2 = [w_2, t_2]$, then we may choose $w_3 \in xO_f \cap O_f$ and $w_4 \in y^{-1}O_f \cap O_f$ such that $w_2 = w_3 + w_4$. Moreover, since the map $U(L)_f \to O_f$ which takes $[w,t] \to w$ is surjective, there exist $u_3 \in U(xL)_f$ and $u_r \in U(y^{-1}L)_f$ such that $u_4^{-1}u_3^{-1}u_2 \in \mathcal{N}(L)_f$, and this proves (a).

We will now prove (b). Observe that by (a) it will suffice to show that if p is a prime which splits in K, $p = \pi\overline{\pi}$, p is relatively prime to ν, and $x = \pi/\overline{\pi}$ then $l(x^{-1})(l(x)\theta) = p^{-1}\theta$. By definition, we have

$$l(x^{-1})(l(x)\theta) = \int_{U(L)_f}\int_{U(L)_f}\theta(mu_1x^{-1}u_2x)du_2du_1$$

$$= \int_{U(L)_f}\left(\int_{U(L)_f}\theta(mu_1x^{-1}u_2x)du_1\right)du_2.$$

Let $u_1 = [w_1, t_1]$ and $u_2 = [w_2, t_2]$, and let $\alpha(w_1, w_2) \in \mathbf{Q}_p$ be defined by

$$u_1x^{-1}u_2xu_1^{-1} = [w_2/x, t_2 + \alpha(w_1, w_2)],$$

then

$$\alpha(w_1, w_2) = \overline{w}_1x^{-1}w_2\tau/2 + \overline{\overline{w}_1x^{-1}w_2\tau/2}.$$

So, if $m = u\beta$ with $u \in U_f$, $\beta \in T_f$, then

$$\int_{U(L)_f} \theta(mu_1x^{-1}u_2x)du_1 = \int_{U(L)_f} \lambda(\nu\zeta(\beta)\alpha(w_1,w_2))du_1\theta(mx^{-1}u_2x),$$

where $\zeta(\beta) = \left(\beta\overline{\beta}/\mathcal{N}(\beta)\right)$. Observe that $[w_1,t_1] \mapsto \lambda(\nu\zeta(\beta)\alpha(w_1,w_2))$ is a character of $U(L)_f$ which is trivial if and only if $w_2 \in \pi O_f$. If this character is not identically 1, then $x^{-1}u_2x \in U(L)_f$, and so $\theta(mx^{-1}u_2x) = \theta(m)$. Since the measure of

$$\{[w_2,t_2] \in U(xL)_f : w_2 \in \pi O_f\}$$

is p^{-1}, we have $l(x^{-1})(l(x)\theta)(m) = p^{-1}\theta(m)$. **Q.E.D.**

4.2. Shintani's representation

In this section, we deduce some consequences of Proposition 4.1 which allow us to compute the composition of any two Shintani operators.

Let I_ν^1 denote the group of ideals of K which are prime to ν and which have norm 1, and for $x \in I_\nu^1$, let $num(x)$ be the numerator of x. By Proposition 4.1(b), we see that the map

$$(l_1(x)\theta) = \mathcal{N}(num(x))^{1/2}\,(l(x)\theta)\,,$$

defines a unitary representation of I_ν^1 on $V_{k,\nu}(L)$. This representation decomposes into a direct sum of unitary characters κ of I_ν^1, so if we let

$$V_{k,\nu}(L)_\kappa = \left\{\theta : l_1(x)\theta = \kappa(x)\theta, \forall x \in I_\nu^1\right\},$$

then $V_{k,\nu}(L) = \bigoplus_\kappa V_{k,\nu}(L)_\kappa$ where the sum is an orthogonal direct sum.

The representation l_1 also induces a representation on the subspace of primitive theta functions. Indeed, let $a \in I_\nu^1$, and let θ be a primitive, then to show that $l(a)\theta$ is primitive, we must show that for every integral ideal b such that $\mathcal{N}(b)|\nu$ we have $l(b^{-1})l(a)\theta = 0$. But since the ideals b and a are relatively prime and since $\mathcal{N}(a)\nu$ and $\mathcal{N}(b^{-1})\nu$ are integral ideals, Proposition 4.1(a) implies that $l(a)$ and $l(b^{-1})$ commute when operating on $V_{k,\nu}(L)$. Thus, since θ is primitive we have $l(b^{-1})\theta = 0$ which implies that $l(b^{-1})l(a)\theta = 0$ and so $l(a)\theta$ is primitive. Observe also that if we put $\kappa(x) = 0$ for ideals x which have norm 1 and are not relatively prime to ν, then for primitive theta functions $\theta \in V_{k,\nu}(L)_\kappa^0$ the relation $l(x)\theta = \kappa(x)\theta$ holds for all ideals x with norm 1.

We now prove a Lemma which will allow us, in principle, to calculate $l(x)l(y)\theta$ using the characters κ. Since $V_{k,\nu}(L)$ is spanned by theta functions of the form $l(a)\theta'$ where a is an integral ideal and θ' is primitive theta function in $V_{k,\nu'}(L)^0$ for integers $\nu' = \nu/\mathcal{N}(a)$, we may assume that θ has this form. The next lemma will show that for any $x \in I_K$, $l(x)\theta$ is either 0 or is again of this form, and this will allow us to compute arbitary compositions of the Shintani operators:

Lemma 4.2. *Let $\theta \in V_{k,\nu}(L)^0_\kappa$, $\theta \neq 0$. We have the following:*

a) *Let x, a be ideals of K such that a is an integral ideal, and let $ax = b/c$ where b, c are relatively prime integral ideals, then*

$$l(x)\,(l(a)\theta) = \begin{cases} 0 & \text{if } \bar{c} \text{ does not divide } b, \\ \mathcal{N}(c)^{-1/2}\kappa(c/\bar{c})l(b/\bar{c})\theta & \text{if } \bar{c}|b. \end{cases}$$

Note that, in the second case, if $c/\bar{c}$ is not relatively prime to ν then $\kappa(c/\bar{c}) = 0$ and so $l(x)l(a)\theta = 0$.

b) *Let κ^* be the character of the group of ideals prime to νD defined by*

$$\kappa^*(x) = \kappa(x/\bar{x}) \prod_{p\ inert} (-1)^{ord_p(x)},$$

then κ^* is the map on ideals induced by a unitary Hecke character of K, which we also denote by κ^*, with weight $-(2k-1)$ and conductor C_ν which divides $4\nu D$. In the notation of §1.6, we have $\kappa^* \in \mathcal{H}^*_{-(2k-1),C_\nu}(K)$.

Proof: The first part of the lemma follows easily from Proposition 4.1(a) and the definition of κ. Indeed, since the denominator of a is 1, we have $l(x)l(a)\theta = l(xa)\theta = l(b/c)\theta$.

If $\bar{c}$ does not divide b, then $\mathcal{N}(b/c)$ is not an integral ideal and there are two cases. First, if $\nu\mathcal{N}(b/c)$ is not an integral ideal, then $V_{k,\nu\mathcal{N}(b/c)}(L) = \{0\}$, so $l(xa)\theta = 0$. Second, if $\nu\mathcal{N}(b/c)$ is an integral ideal, then c can be factored as c_1c_2 where c_1 and c_2 are relatively prime and $\mathcal{N}(c_1)$ is relatively prime to $\mathcal{N}(b)$, and $\mathcal{N}(c_1^{-1})\nu$ is an integral ideal. Thus, by Proposition 4.1(a) $l(b/c)\theta = l(b/c_2)l(1/c_1)\theta = 0$.

So assume that $\bar{c}$ divides b, and write $b = b_1\bar{c}$, then

$$l(b/c)\theta = l(b_1)l(\bar{c}/c)\theta = \mathcal{N}(c)^{-1/2}\kappa(\bar{c}/c)l(b_1)\theta.$$

Since $\kappa(\bar{c}/c) = 0$ if $\bar{c}/c$ is not relatively prime to ν, this proves part (a).

Part (b) of this lemma is a fairly deep result which we do not use in the proof of our main theorem. Indeed, all we need to know about κ^* for our main theorem is that it is a unitary character of the group of ideals prime to νD, which is clear. The problem of determining just which κ^* appear as eigencharacters of Shintani's representation is known explicitly only for $K = \mathbf{Q}(\sqrt{-1})$. The only place in which we need more information about κ is when we derive the arithmeticity of the Fourier-Jacobi coefficients of Eisenstein series from our main theorem. To make this derivation we need to know, in Lemma 8.4, that κ^* is a Hecke character with conductor, but no information is needed about the conductor. We refer the reader to Shintani's article for a complete proof of part (b) of this Lemma [13, Prop. 3]. **Q.E.D.**

4.3. An action at the ramified places

In this section we define another commuting family of endomorphisms on the space of theta functions. These endomorphisms are projections associated to the ramified primes of K. These projections commute with the operators $l(x)$ when x has norm 1, and can be used to further decompose the eigenspaces $V_{k,\nu}(L)_\kappa$ into a direct sum of intrinsically defined orthogonal subspaces.

Let p be a prime which ramifies in K, and let π be a generator of the prime ideal of O_p. Define

$$U(L)_p^* = \{[w,t] : w \in \pi^{-1}O_p, t \in \mathbf{Q}_p, t + w\overline{w}\tau/2 \in \pi^{-1}O_p\}.$$

This is a subgroup of $U(\pi^{-1}L)_p$ which contains $U(L)_p$. It is easy to see that the projection map $[w,t] \to w$ maps $U(L)_p^*$ onto $\pi^{-1}O_p$, and that the kernel of the projection map is $N(L)_p$. Thus, the index of $U(L)_p$ in $U(L)_p^*$ is p.

Let l_π be the linear transformation of $V_{k,\nu}(L)$ defined by

$$[l_\pi\theta](m) = p^{-1}\int_{U(L)_p^*} \theta(mh)dh;$$

then $l_\pi^2 = l_\pi$, so l_π has two eigenvalues, 0 and 1 and hence l_π is a projection of $V_{k,\nu}(L)$ onto the subspace $l_\pi V_{k,\nu}(L)$. Moreover, it is easy to see that for all $\theta_1, \theta_2 \in V_{k,\nu}(L)$ we have

$$(l_\pi\theta_1, \theta_2) = (\theta_1, l_\pi\theta_2).$$

Thus, the kernel of l_π is the orthogonal complement of its image. So for each ramified prime π we obtain an orthogonal direct sum decomposition

$$V_{k,\nu}(L) = image(l_\pi) \oplus kernel(l_\pi).$$

It is easy to see that the operators $\{l_\pi : \pi \text{ ramifies in } K\}$ commute, we can decompose $V_{k,\nu}(L)$ into an orthogonal direct sum of the subspaces consisting of simultaneous eigenfunctions of this family of operators. We now introduce some notation for these subspaces. For a subset Σ of the set of primes of $\mathbf{Q}$ ramifying in K, let $V_{k,\nu}(L)[\Sigma]$ be the subspace of simultaneous eigenfunctions of the operators $\{l_\pi\}$ whose eigenvalues are 1 for π in Σ and 0 for π not in S:

$$V_{k,\nu}(L)[\Sigma] = \bigcap_{\pi\in\Sigma} image(l_\pi) \cap \bigcap_{\pi\notin\Sigma} kernel(l_\pi).$$

A straightforward calculation shows that the operators l_π and $l(x)$ commute if x has norm 1, and that l_π maps primitive theta functions to primitive theta functions. So we can further decompose $V_{k,\nu}(L)$ into simultaneous eigenspaces of both families of operators. For a character κ occuring in the decomposition of Proposition 4.1, let

$$V_{k,\nu}(L)_{\kappa,\Sigma} = V_{k,\nu}(L)_\kappa \cap V_{k,\nu}(L)[\Sigma],$$

then $V_{k,\nu}(L) = \bigoplus_{\kappa,\Sigma} V_{k,\nu}(L)_{\kappa,\Sigma}$ is an orthogonal direct sum decomposition. Since the operators l_π map primitive theta functions to primitive theta functions, we get a similar decomposition of that space into subspaces, and we will let Ξ denote the set of all triples $\xi = (\nu, \kappa^*, \Sigma)$ such that the space V_ξ

$$V_\xi = V_{k,\nu}(L)_\kappa^0 \cap V_{k,\nu}(L)[\Sigma].$$

Thus, for every triple $\xi = (\nu, \kappa^*, \Sigma)$ in Ξ, the eigenspace V_ξ is the space of all theta functions θ of level ν and weight k for L such that

i) $l(a^{-1})\theta = 0$ for all integral ideals a such that $\mathcal{N}(a)$ divides ν;

ii) $l(x)\theta = \mathcal{N}(num(x))^{-1/2}\kappa(x)$ for all norm 1 ideals of K which are prime to ν; and

iii) $l_\pi\theta$ is 1 if the ramified prime π is in the set Σ and is 0 if π is not in Σ. The space V_ξ for $\xi = (\nu, \kappa, \Sigma)$ will sometimes be denoted $V_{k,\nu}(L)^0_{\kappa,\Sigma}$ when we wish to emphasize the character κ and the set Σ. We formulate the observations of this section into a Lemma:

Lemma 4.3.1. *[13, p.40] Let Ξ be the set of triples for which V_ξ is nonzero, and let Ξ_ν denote the subset of those triples whose first element is ν: $\xi = (\nu, \kappa^*, \Sigma)$. Then*

$$V_{k,\nu}(L)^0 = \bigoplus_{\xi \in \Xi_\nu} V_\xi,$$

and the sum is orthogonal direct.

To complete our description of the action of the operators $\{l_\pi\}$ on $V_{k,\nu}(L)$, we need to describe how they act on the subspaces of oldforms. This is the content of the next lemma.

Lemma 4.3.2. *[13, p. 70] Let $\xi = (\nu, \kappa^*, \Sigma) \in \Xi_\nu$ and let $\theta \in V_\xi$. Let a be an integral ideal of K, and let π be a ramified prime of K. Then*

$$l_\pi l(a)\theta = \begin{cases} l(a)\theta & \text{if } \pi \in \Sigma \text{ or } \pi | a; \\ 0 & \text{if } \pi \notin \Sigma \text{ and } \pi \not| a . \end{cases}$$

Proof: This is a straightforward computation.

4.4. A system of coordinates for modular forms

In this section we show how a basis of the spaces of primitive theta functions can be used to construct a basis for the graded ring of theta functions. We will use this basis to construct a system of complex number coordinates for modular forms, as mentioned in the beginning of this chapter.

Recall, from the previous section that $\mathcal{B}$ denotes a basis of the space of primitive theta functions, and that we have specified that this basis should be the union of bases $\mathcal{B}_\xi$ for the eigenspaces V_ξ as ξ ranges over the set of eigenvalues Ξ. We have seen that the graded ring of theta functions is spanned by the images $l(a)\theta$ of these basis elements as a ranges over integral ideals of K. In this section we show that these elements are actually linearly independent and so form a basis for the graded ring of theta functions. More precisely, we will give Shintani's proof of the following lemma:

Lemma 4.4. *Let Ξ be as above, then the set*

$$\{l(a)\theta : \theta \in \mathcal{B}_\xi, \xi \in \Xi, a \text{ an integral ideal}\}$$

is a basis for the graded ring of theta functions and so

$$\bigoplus_{\nu} V_{k,\nu}(L) = \bigoplus_{\xi\in\Xi} \bigoplus_{\theta\in\mathcal{B}_\xi} \bigoplus_{a} \mathbf{C}l(a)\theta,$$

where the outer two sums are orthogonal direct, but the inner sum is only direct.

This will be proved in a sequence of lemmas. We will first show that the outer sum is orthogonal, then show that the middle sum is orthogonal and the inner sum is direct. Recall that the Siegel inner product on the graded ring of theta functions is defined so that the subspaces $\{V_{k,\nu}(L) : \nu \geq 0\}$ are orthogonal.

Lemma 4.4.1. *For $\xi \in \Xi$, let $W_\xi = \sum_a l(a)V_\xi$ where the sum is over all integral ideals of K. Let $\xi_1, \xi_2 \in \Xi$, $\xi_1 \neq \xi_2$. Then W_{ξ_1} and W_{ξ_2} are orthogonal subspaces of the graded ring of adelic theta functions.*

Proof: We must show for all integral ideals a, b and all $\theta_1 \in V_{\xi_1}$, $\theta_2 \in V_{\xi_2}$, that

$$(l(a)\theta_1, l(b)\theta_2) = 0.$$

Let $\xi_1 = (\nu_1, \kappa_1^*, \Sigma_1)$, $\xi_2 = (\nu_2, \kappa_2^*, \Sigma_2)$. Since $l(a)\theta_1$ has level $\mathcal{N}(a)\nu_1$ and and $l(b)\theta_2$ has level $\mathcal{N}(b)\nu_2$, we may assume that these two levels are equal. Moreover, since the operator $l(a)$ is adjoint to $l(a^{-1})$ (c.f. §4.1), we have

$$(l(a)\theta_1, l(b)\theta_2) = (\theta_1, l(a^{-1})l(b)\theta_2),$$

and we can evaluate $l(a^{-1})l(b)\theta_2$ using Lemma 4.2(a). Indeed, let $b/a = b_1/c_1$ where b_1 and c_1 are relatively prime integral ideals. Then by Lemma 4.2(a), the composition $l(a^{-1})l(b)\theta_2$ is zero unless c_1 and ν are relatively prime and $\overline{c}_1|b_1$. So we may assume that c_1 and ν are relatively prime, and that there is an integral ideal c_2 such that $\overline{c}_1c_2 = b_1$. In this case, Lemma 4.2(a) asserts that

$$l(a^{-1})l(b)\theta_2 = \mathcal{N}(c_1)^{-1}\kappa_2(c_1/\overline{c}_1)l(c_2)\theta.$$

If $c_2 \neq 1$ then

$$(l(a)\theta_1, l(b)\theta_2) = \mathcal{N}(c_1)^{-1}\kappa_2(c_1/\overline{c}_1)(\theta_1, l(c_2)\theta_2),$$

and this is zero since θ_1 is primitive. If $c_2 = 1$, then

$$(l(a)\theta_1, l(b)\theta_2) = \mathcal{N}(c_1)^{-1}\kappa_2(c_1/\overline{c}_1)(\theta_1, \theta_2),$$

and this is zero because the spaces V_{ξ_1} and V_{ξ_2} are orthogonal. **Q.E.D.**

Lemma 4.4.2. *Let $\xi \in \Xi$, let $\mathcal{B}_\xi$ be a basis for V_ξ, and let $W_\theta = \sum_a \mathbf{C}\,(l(a)\theta)$. Then the subspaces W_θ, $W_{\theta'}$ are orthogonal for all $\theta \neq \theta'$ in $\mathcal{B}_\xi$ and for each $\theta \in V_\xi$, the set*

$$\{l(a)\theta : a\,\text{integral}\}$$

is a basis for W_θ. Thus if W_ξ is as in Lemma 4.4.1, we have

$$W_\xi = \bigoplus_{\theta \in \mathcal{B}_\xi} \bigoplus_{a} l(a)\theta,$$

where the outer sum is orthogonal and the inner is direct.

Proof: Since $\mathcal{B}_\xi$ consists of primitive orthogonal elements, it is clear for any $\theta_1 \neq \theta_2$ in $\mathcal{B}_\xi$ and any integral ideals a, b that

$$(l(a)\theta_1, l(b)\theta_2) = (\theta_1, l(a)^{-1} l(b)\theta_2) = 0.$$

Indeed, the proof of the previous lemma shows that the inner product on the right hand side is either zero or a constant multiple of the inner product of θ_1 and θ_2, which is zero by assumption.

Thus, we must prove for any $\theta \in V_\xi$ that the set

$$\{l(a)\theta : a \,\text{integral}\},$$

consists of linearly independent elements. To prove this it will suffice to show that the subsets of elements of the same level are linearly independent. So, let N be an integer which is the norm of some ideal in K and let $\{a_i\}$ be the set of all integral ideals of K whose norm is N. It will suffice to show that the matrix $A(N) = (A_{ij}(N))$ defined by

$$A_{ij}(N) = \frac{(l(a_i)\theta, l(a_j)\theta)}{(\theta, \theta)} = \frac{(\theta, l(a_i^{-1})l(a_j)\theta)}{(\theta, \theta)},$$

is nondegenerate. Indeed, we will show that its determinant is

$$det(A(N)) = \prod_{p|N}{}' \left(1 - p^{-1}\right)^{ord_p(N)},$$

where the product is over all primes which split in K, are relatively prime to ν, and divide N. By Lemma 4.2(a), we see that

$$A_{ij}(N) = \kappa(a_j/a_i)/\mathcal{N}(num(a_j/a_i))^{1/2},$$

where $\kappa(c) = 0$ unless c is relatively prime to ν. Let $\{p_j : j = 1, \ldots, n\}$ be the set of primes dividing N which split in K, and let $N = \prod_j p_j^{e_j} N'$ where N' is not divisible by any split primes. Then it is easy to see that the matrix $A(N)$ is the Kronecker product (i.e. tensor product) of the matrices $A(p_j^{e_j})$ and $A(N')$. Moreover, since there is only one integral ideal of K whose norm is N', we see that $A(N') = (1)$ (a 1 by 1 matrix), and so $det(A(N')) = 1$. It is also clear that $det(A(p_j^{e_j})) = 1$ for all split primes p_j dividing ν. For the remaining primes, the matrix $A(p^e)$ has the form

$$A(p^e) = \begin{pmatrix} 1 & x & x^2 & \ldots & x^{e-1} & x^e \\ y & 1 & x & \ldots & x^{e-2} & x^{e-1} \\ y^2 & y & 1 & \ldots & x^{e-3} & x^{e-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ y^{e-1} & y^{e-2} & y^{e-3} & \ldots & 1 & x \\ y^e & y^{e-1} & y^{e-2} & \ldots & y & 1 \end{pmatrix}.$$

where $x = \kappa(\pi/\overline{\pi})/p^{1/2}$ and $y = \overline{\kappa(\pi/\overline{\pi})/p^{1/2}}$. This matrix has determinant $(1-xy)^e$ as can easily be seen by expanding by minors along the first row and observing that all but the first two minors vanish. Since $xy = p^{-1}$ in our case, this proves that

$$det(A(N)) = \prod_p{}' \left(1 - p^{-1}\right)^{ord_p(N)}.$$

Q.E.D.

4.5. Hecke operators on modular forms

We will see in Chapter 7 that the Eisenstein series E_Λ are eigenfunctions of the ring of Hecke operators on $A_k(L,\chi)$. In the rest of this Chapter, we describe Shintani's results on the generators of this ring, their effect on the adelic theta coefficients of modular forms, and then we deduce the consequences for simultaneous eigenfunctions.

In this section, we define the ring of Hecke operators on $A_k(L,\chi)$ following Satake [9]. These results, and the next lemma, follow immediately from general results in the theory of spherical functions on reductive algebraic groups over p-adic fields [9,§9].

Let p be a rational prime, and let $C_0(G_p, G(L)_p)$ be the space of continuous complex-valued functions with compact support on $G(L)_p\backslash G_p/G(L)_p$. This is a commutative subring of the group ring of G_p. Let dg be Haar measure on G_p, normalized so that $G(L)_p$ has measure 1. For each $\phi \in C_0(G_p, G(L)_p)$, define a linear operator, T_ϕ, on $A_k(L,\chi)$ by

$$[T_\phi(f)](g) = \int_{G_p} f(gx)\phi(x)dx.$$

Then the map $\phi \mapsto T_\phi$ defines a representation of the commutative algebra $C_0(G_p, G(L)_p)$ in $A_k(L,\chi)$, let $R_p(L)$ be the image of this representation in the endomorphism ring of $A_k(L,\chi)$.

Lemma 4.5. *[9] Let p be a rational prime, and let π be a prime of K dividing p, then the ring $R_p(L)$ is generated by T_π and $T_{\overline{\pi}}$, where T_π is an abbreviation for the operator T_{ϕ_π} associated to the characteristic function ϕ_π of the subset $S(\pi)$ defined as follows:*

a) *If p is inert in K, then $\pi = \overline{\pi}$ and*

$$S(\pi) = \{g \in G_p : gL_p \subset L_p, ord_p(\mu(g)) = \mathcal{N}(\pi)\};$$

b) *if p is ramified in K, then $\pi = \overline{\pi}$ and*

$$S(\pi) = \{g \in G_p : gL_p \subset L_p, ord_p(\mu(g)) = \mathcal{N}(\pi)\};$$

c) *if p is split in K, then there exist idempotents $e, \overline{e}$ in K_p such that $e\overline{e} = 0$ and $e + \overline{e} = 1$, so $L_p \cong eL_p \oplus \overline{e}L_p$, and if $g \in G_p$, then $g(eL_p) \subset eL_p$. Let $\epsilon(g)$ be the set of elementary divisors of g with respect to eL_p, and let $\pi = pe + \overline{e}$; then*

$$S(\pi) = \{g \in G_p : gL_p \subset L_p, ord_p(\mu(g)) = \mathcal{N}(\pi), \epsilon(g) = \{1, 1, \pi\}\}.$$

Corollary. *Assumptions being as in the Lemma, the sets $S(\pi)$ are given more explicitly as follows:*

a) *If p is inert in K, then*

$$S(\pi) = U(L)_p d(\pi, \pi^{-1}) G(L)_p \cup U(L)_p N(p^{-1}L) d(\pi, 1) G(L)_p \cup U(L)_p d(\pi, \pi) G(L)_p;$$

b) *if p is ramified in K, and π generates the prime ideal of O_p, then*

$$S(\pi) = U(L)_p d(\pi, \pi^{-1}) G(L)_p \cup U(L)_p^* d(\pi, 1) G(L)_p \cup U(L)_p d(\pi, \pi) G(L)_p;$$

c) *if p is split in K and $\pi = pe + \overline{e}$, then*

$$S(\pi) = U(L)_p d(\overline{\pi}, \pi^{-1}) G(L)_p \cup U(L)_p d(\pi, \pi/\overline{\pi}) G(L)_p \cup U(L)_p d(\overline{\pi}, \overline{\pi}) G(L)_p.$$

Proof: First, recall that there is a local Iwasawa decomposition $G_p = P_p G(L)_p$ of G_p with respect to the parabolic subgroup P_p and the maximal compact subgroup $G(L)_p$ (This has been mentioned before and will be proved in Chapter 5). Since $S(\pi)$ is a union of double $G(L)_p$-cosets, the Iwasawa decomposition implies that there exists a set of elements $\{g_i\} \subset P_p/P(L)_p$ such that

$$S(\pi) = \bigcup_i U(L)_p g_i G(L)_p,$$

where the union is disjoint. Moreover, each of the g_i has the form

$$[w, t + w\overline{w}D/2] d(z, x) = \begin{pmatrix} z\overline{x} & z\tau\overline{w} & (z/x)(t + w\overline{w}\omega) \\ 0 & z & (z/x)w \\ 0 & 0 & (z/x) \end{pmatrix}.$$

where $\tau = \sqrt{-D}$, $\omega = (D+\tau)/2$ generates O_p over $\mathbf{Z}_p$, $w \in K_p$ $t \in \mathbf{Q}_p$, $z, x \in K_p^*$ and each g_i is subject to the constraints stated in Lemma 4.5 (c.f. §1.2 for the definitions of $[w,t]$ and $d(z,x)$). This corollary can be proved by simply translating Satake's conditions (in Lemma 4.5) into more explicit conditions on the elements $[w, t + w\overline{w}D/2] d(z, x) \in S(\pi)$.

In particular, since $g_i L_p = L_p$ and $L_p = O_p^3$, we see that $g_i \in M_3(O_p)$, which implies that z, $z\overline{x}$, and z/x are in O_p. Moreover, since

$$\mu([w, t + w\overline{w}D/2] d(z, x)) = z\overline{z},$$

we see that z is either π or $\overline{\pi}$, up to units.

If p is inert or ramified, then π is the only prime of K dividing p, and we see that x must be in the set $\{\pi^{-1}, 1, \overline{\pi}\}$. If p is split however there are two cases:

a) if $z = \pi$, then $x \in \{\pi, \overline{\pi}^{-1}, \pi/\overline{\pi}\}$

b) if $z = \overline{\pi}$, then $x \in \{\overline{\pi}, \pi^{-1}, \overline{\pi}/\pi\}$

In the split case, the elementary divisors of all elements g of $S(\pi)$ are required to be $\{1, 1, p\}$. Since $G_p = G(L)_p D_p G(L)_p$, this is a condition on the torus element $d(z, x)$ and it is easily seen that the only remaining possibilities for $d(z, x)$ are $\left\{d(\overline{\pi}, \pi^{-1}), d(\pi, \pi/\overline{\pi}), d(\overline{\pi}, \overline{\pi})\right\}$.

To complete the proof of the corollary, we need only show that the conditions placed on w and t by the requirement that

$$[w, t + w\overline{w}D/2]d(z, x) \in M_3(O_p),$$

are precisely those stated in the Corollary, and this is a simple calculation which we leave to the reader. **Q.E.D.**

4.6. The action of Hecke operators on adelic theta coefficients

In this section, we will extend the action of the ring of Hecke operators to the space $C(M_p/M(L)_p)$ of continous complex valued functions on $M_p/M(L)_p$ in such a way that

$$(T_\phi F)\,|M_{\mathbf{A}} = T_\phi\,(F|M_{\mathbf{A}}) \qquad \forall F \in A_k(L, \chi), \quad \phi \in C_0(G_p, G(L)_p.$$

We also show that this extended action of the Hecke operators can be expressed as a linear combination of the Shintani operators $l(x)$. From these explicit linear combinations, it is straightforward to derive an explicit formula for the action the Hecke operators induce on the adelic theta coefficients of modular forms.

First we must introduce some maps that make the Iwasawa decomposition of G_p explicit. Define a map $\Delta : G_p/G(L)_p \to P_p/P(L)_p$ by the requirement that $g \in \Delta(g)G(L)_p$. Since P_p is the product of its maximal torus $D_p = Z_p \times T_p$ and its unipotent radical U_p, we can also define maps Δ_Z, Δ_M, Δ_T, Δ_U such that

$$g \in \Delta_Z(g)\Delta_M(g)G(L)_p \qquad \Delta_M(g) = \Delta_T(p)\Delta_U(p),$$

where Δ_X maps $G_p/G(L)_p$ into $X_p/X(L)_p$ for each $X = Z, T, U, M$. These maps are all formed by composing projections with Δ. We have already used the map $\Delta_T = \delta$ in Chapter 2 to define Eisenstein series.

We can now define an action of the Hecke operators T_ϕ on the space of continuous functions on $M_p/M(L)_p$ by

$$(T_\phi F)\,(m) = \int_{G_p} \chi(\Delta_Z(x))F(m\Delta_M(x))\phi(x)dx,$$

for all $\phi \in C_0(G_p, G(L)_p)$, and $F \in C(M_p/M(L)_p)$. If $F \in A_k(L, \chi)$ and $g \in G_p$ then the right-$G(L)_p$ invariance of F implies that

$$F(mg) = \chi(\Delta_Z(g))\; F(m\,\Delta_M(g)),$$

from which it follows that $(T_\phi F)\,|M_{\mathbf{A}} = T_\phi\,(F|M_{\mathbf{A}})$. Thus, if $F|M_{\mathbf{A}} = \sum_\nu F_\nu$ is the adelic theta expansion of F we see that the action of T_ϕ on F can be described by an action on its adelic theta coefficients:

$$(T_\phi F)(m) = \sum_\nu (T_\phi F_\nu)\,(m),$$

provided we show that the Hecke operators map the graded ring of adelic theta functions to itself. Indeed, we will see that the generators T_π actually can be expressed in terms of the Shintani operators $l(x)$ defined above.

Lemma 4.6. *[13,p. 33] Let p be a rational prime, let π be a prime of K dividing p, and let $F \in V_{k,\nu}(L)$. Then*

a) if p is inert in K, $T_\pi F =$

$$\chi(\pi)\, l(\pi)F + \delta_{p,\nu}\, \chi(\pi)p\, F + \chi(\pi)p^4\, l(\pi^{-1})F,$$

where $\delta_{p,\nu}$ is 1 if $p|\nu$ and 0 otherwise;

b) if p ramifies in K, then $T_\pi(F) =$

$$\chi(\pi)\, l(\pi)F + \chi(\pi)p\, l_\pi F + \chi(\pi)p^2\, l(\pi^{-1})F;$$

c) if p splits in K, then $T_\pi(F) =$

$$\chi(\overline{\pi})\, l(\pi)F + \chi(\pi)p\, l(\overline{\pi}/\pi)F + \chi(\overline{\pi})p^2\, l(\overline{\pi}^{-1})F.$$

Proof: Notice that each of the sets $S(\pi)$ defined in the Corollary to Lemma 4.5 is a union of three sets of the form $S = R_p d(z,x)G(L)_p$ where R_p is a compact subgroup of U_p containing $U(L)_p$. Let T_S be the operator defined by

$$(T_S F)(m) = \int_S \chi(\Delta_Z(g))F(m\Delta_M(g))dg \qquad \forall F \in C(M_p/M(L)_p).$$

We will now show that

$$(T_S F)(m) = (mod(x)\alpha(x))^{-1}\, \chi(z) \int_{R_p} F(mux)du,$$

where $\alpha(x)$ is the measure of $x^{-1}R_p x \cap U(L)_p$ and $mod(x)$ is the measure of $xU(L)_p$ with respect to the right-invariant Haar measure du on U_p. Indeed, since the Haar measure dg on G_p is left and right invariant, we can make the change of variables $m' = d(z,x)m$ in the integral defining T_S which replaces the domain of integration by $x^{-1}R_p x G(L)_p$ and so

$$(T_S F)(m) = \chi(z) \int_{x^{-1}R_p x G(L)_p} F(mx\Delta_M(g))dg.$$

Since R_p is a compact subgroup containing $U(L)_p$, this last integral is simply the sum of the right translates of the integrand $F(mxu_i)$ by a complete set of representatives for the finitely many cosets $x^{-1}R_p x G(L)_p/G(L)_p$, which we may take to be elements of $x^{-1}R_p x$. These coset representatives are also a complete set of representatives for the cosets $x^{-1}R_p x U(L)_p/U(L)_p$, since $U(L)_p$ has measure 1 with respect to du. Moreover, these coset representatives are a complete set of representatives for the right cosets of $x^{-1}R_p x \cap U(L)_p$ in $x^{-1}R_p x$ and since each of these cosets has measure $\alpha(x)$ we can express the previous integral as an integral over the domain in $x^{-1}R_p x$:

$$(T_S F)(m) = \alpha(x)^{-1}\chi(z) \int_{xR_p x^{-1}} F(mxv)dv,$$

where $\alpha(x)$ is the measure of $x^{-1}R_p x \cap U(L)_p$ and we have made use of the fact that $\Delta_M|(M_p/M(L)_p)$ is the identity map. Finally, we make a change of variables $u = xvx^{-1}$, $du = mod(x)dv$ and we obtain the formula claimed:

$$(T_S F)(m) = mod(x)^{-1}\alpha(x)^{-1}\chi(z)\int_{R_p} F(mux)du.$$

By the Corollary to Lemma 4.5, the set R_p is either $U(L)_p$, $U(L)_p^*$, or $N(p^{-1}L)_p U(L)_p$ and in the first two cases we have, by definition (c.f. §4.1, §4.3)

$$\left(l(x^{-1})F\right) = \int_{U(L)_p} F(mux)du \qquad \text{and} \qquad pl_\pi = \int_{U(L)_p^*} F(mu)du.$$

In the last case, if $F \in V_{k,\nu}(L)$, and $m = av$ for some $a \in T_p$, $v \in U_p$, then we have

$$\int_{U(L)_p N(p^{-1}L)_p} F(avu)du = \sum_{t=0}^{p-1} F(av[0,t/p]) =$$

$$\sum_{t=0}^{p-1} \lambda_p(\nu a\overline{a}t/\mathcal{N}(a))\; F(av[0,t/p]) = \begin{cases} pF(av) & \text{if } p|\nu, \\ 0 & \text{otherwise.} \end{cases}$$

To conclude the lemma we need only observe that $mod(x) = \|x\|^2$ and we need to calculate $\alpha(x)$, which is the measure of $x^{-1}R_p x \cap U(L)_p$ for each of these three sets R_p, and for the three possible values π^{-1}, $\overline{\pi}/\pi$, and $\overline{\pi}$ of x. Consider first the case $R_p = U(L)_p$, then if $x = \pi^{-1}$,

$$\alpha(\pi^{-1})^{-1} = [U(L)_p : U(L)_p \cap U(\pi L)_p] = [U(L)_p : U(\pi L)_p] = mod(\pi)^{-1},$$

and so

$$\left(mod(\pi^{-1})\alpha(\pi^{-1})\right)^{-1} = 1.$$

If $x = \overline{\pi}/\pi$, and $\pi \neq \overline{\pi}$ then $mod(x) = 1$ and

$$\alpha(\overline{\pi}/\pi)^{-1} = [U(L)_p : U(L)_p \cap U(\pi/\overline{\pi}L)_p] = [U(\overline{\pi}L)_p : U(\overline{\pi}L)_p \cap U(\pi L)_p] = p,$$

and so

$$\left(\alpha(\overline{\pi}/\pi)mod(\overline{\pi}/\pi)\right)^{-1} = p.$$

Finally, if $x = \overline{\pi}$, then $mod(x) = \mathcal{N}(\pi)^{-2}$ and

$$\alpha(\overline{\pi})^{-1} = [U(L)_p : U(L)_p \cap U(\overline{\pi}^{-1}L)_p] = 1,$$

and so

$$\left(\alpha(\overline{\pi})mod(\overline{\pi})\right)^{-1} = \mathcal{N}(\pi)^2.$$

The only remaining cases are when $x = 1$ and R_p is $U(L)_p^*$ or $N(p^{-1}L)_p U(L)_p$. In these cases, $mod(x) = 1$ and $R_p \supset U(L)_p$ so $\alpha(1) = 1$. This proves the Lemma. **Q.E.D.**

As an immediate corollary we can see how the Hecke operators act on the adelic theta coefficients of a modular form F.

Corollary. *[13,p. 33] Let p be a rational prime and let π be a prime of K dividing p. Let $F \in A_k(L,\chi)$ and let $F = \sum_\nu F_\nu$ be its adelic theta expansion, then*
a) if p is inert in K, $(T_\pi(F))_\nu =$

$$\chi(\pi)\, l(\pi) F_{\nu/p} + \delta_{p,\nu}\, \chi(\pi) p\, F_\nu + \chi(\pi) p^4\, l(\pi^{-1}) F_{\nu p},$$

where $\delta_{p,\nu}$ is 1 if $p|\nu$ and 0 otherwise;
b) if p ramifies in K, then $(T_\pi(F))_\nu =$

$$\chi(\pi)\, l(\pi) F_{\nu/p} + \chi(\pi) p\, l_\pi F_\nu + \chi(\pi) p^2\, l(\pi^{-1}) F_{\nu p};$$

c) if p splits in K, then $(T_\pi(F))_\nu =$

$$\chi(\overline{\pi})\, l(\pi) F_{\nu/p} + \chi(\pi) p\, l(\overline{\pi}/\pi) F_\nu + \chi(\overline{\pi}) p^2\, l(\overline{\pi}^{-1}) F_{\nu p}.$$

4.7. Formal Dirichlet series

In this section we define certain powerseries in infinitely many variables $\{x_\pi : \pi \,\text{prime}\}$ associated to a modular form F in $A_k(L,\chi)$ (c.f. §2.1), we call these the formal Dirichlet series associated to F.

Let $F \in A_k(L,\chi)$ and let $F = \sum_\nu F_\nu$ be its adelic theta expansion as in §2.4, then the adelic theta coefficients are adelic theta functions, indeed, $F_\nu \in V_{k,\nu}(L)$ (c.f. §2.6). Let Ξ be the set of eigenvalues of the Shintani operators and for each $\xi \in \Xi$ let V_ξ be the associated eigenspace, as in §4.4. Further, let $\mathcal{B}_\xi$ be an orthonormal basis of V_ξ for each ξ and let $\mathcal{B}$ be the union of the bases $\mathcal{B}_\xi$. Then we have seen that

$$\{l(a)\theta : \theta \in \mathcal{B}, a \,\text{integral}\},$$

is a basis for the graded ring $\bigoplus_\nu V_{k,\nu}(L)$ of theta functions. Thus, the adelic theta expansion of F can be written in the form

$$F = \sum_{\theta \in \mathcal{B}} \sum_{a \in I+} c_\theta(a) l(a) \theta,$$

where a ranges over all integral ideals of K and c_θ is a map from the monoid of integral ideals of K into $\mathbf{C}$. This expansion can be written more suggestively as

$$F = \sum_{\theta \in \mathcal{B}} (Z_\theta \cdot \theta) \qquad Z_\theta \cdot \theta = \sum_{a \in I+} c_\theta(a) l(a) \theta,$$

where Z_θ is an operator on the space $\hat{\mathcal{R}} = \prod_\nu V_{k,\nu}(L)$, which we call the formal Dirichlet series associated to F and θ. We call this expansion the Shintani form of the adelic theta expansion of F. We will topologize $\hat{\mathcal{R}}$ by giving it the standard product topology. Observe that the graded ring of adelic theta functions

$$V_{k,*}(L) = \bigoplus_{\nu=0}^{\infty} V_{k,\nu}(L),$$

is dense in $\hat{\mathcal{R}}$ and that $\hat{\mathcal{R}}$ is itself a topological ring containing $V_{k,*}(L)$ as a subring, where multiplication is defined in the obvious way:

$$(x_0, x_1, x_2, \ldots) \cdot (y_0, y_1, y_2, \ldots) = (z_0, z_1, z_2, \ldots),$$

where $z_\nu = \sum_{i+j=\nu} x_i y_j$. A formal Dirichlet series

$$Z = \sum_{a \in I^+} z(a) l(a),$$

operates on an element $x = (x_i) \in \hat{\mathcal{R}}$ by $Z \cdot x = y$ where $y_0 = z(1)x_0$ and for all $i > 0$ we have

$$y_i = \sum_{\substack{a\bar{a}c=i \\ a \in I^+ \\ i \in \mathbf{Z}}} z(a) l(a)(x_i).$$

The formal Dirichlet series Z_θ can be viewed as an element of the formal power series ring $\mathcal{P}$ in the infinite set $\{x_\pi\}$ of variables indexed by the prime ideals of K. Indeed, for any integral ideal a, let x_a denote the monomial in $\mathcal{P}$ given by

$$x_a = \prod_{\pi | a} x_\pi^{ord_\pi(a)}.$$

Then the formal Dirichlet series Z_θ associated to F and θ is defined to be the following element of $\mathcal{P}$:

$$Z_\theta = \sum_{z} c_\theta(a) x_a.$$

We let the ring $\mathcal{P}$ act on $\prod_\nu V_{k,\nu}(L)$ by letting x_π act as $l(\pi)$ and extending by linearity. With this action it is clear that

$$Z_\theta \cdot \theta = \sum_{a} c_\theta(a) l(a) \theta.$$

4.8. Shintani's Eigenfunction Theorem

Shintani's Eigenfunction Theorem asserts that if F is a modular form in $A_k(L, \chi)$ and if

$$F = \sum_{\xi \in \Xi} \sum_{\theta \in \mathcal{B}_\xi} Z_\theta \cdot \theta,$$

is the Shintani form of its adelic theta expansion, then F is a simultaneous eigenfunction of the Hecke operators, with eigenvalues λ if and only if each of the formal Dirichlet series associated to elements $\theta \in \mathcal{B}_\xi$ has the form

$$Z_\theta = c_\theta(1) Z_{\xi,\lambda},$$

where $Z_{\xi,\lambda}$ is a formal Dirichlet series that depends only on ξ, λ, and the character χ. Moreover, this formal Dirichlet series is given by an Euler product of explicit local factors.

$$Z_{\xi,\lambda} = \prod_{p} Z_{\xi,\lambda,p},$$

where the local factors $Z_{\xi,\lambda,p}$ are rational functions in the indeterminates $\{x_\pi : \pi | p\}$.

Theorem 4.7. *[13] Let $F \in A_k(L,\chi)$ be an adelic modular form of weight k and character χ and let*

$$F = \sum_{\xi\in\Xi}\sum_{\theta\in\mathcal{B}_\xi} Z_\theta\cdot\theta, \qquad Z_\theta = \sum_a c_\theta(a)x_a,$$

be the Shintani form of its adelic theta expansion (c.f.§4.7). Then F is a simultaneous eigenfunction of the Hecke operators $\{T_\pi\}$ with eigenvalues $\{\lambda_\pi\}$ (§4.5) if and only if for each ξ and each $\theta\in\mathcal{B}_\xi$, the formal Dirichlet series Z_θ has the form

$$Z_\theta = c_\theta(1)Z_{\xi,\lambda},$$

where $Z_{\xi,\lambda}$ is the formal Dirichlet series explicitly given below by its Euler product. Explicitly, if $\xi=(\nu,\kappa,\Sigma)$ then

$$Z_{\xi,\lambda} = \prod_p Z_{\xi,\lambda,p},$$

where $Z_{\xi,\lambda,p}$ is the rational function in the indeterminates $\{x_\pi : \pi|p\}$ given in the three cases p inert, ramified, split as follows:

a) if p is inert in K, then $Z_{\xi,\lambda,p}(x_p) = R(x_p)/Q(x_p)$, where

$$Q(x) = 1 + \frac{(p-\lambda_\pi\overline{\chi}(\pi))}{p^4}x + \frac{1}{p^4}x^2, \qquad \text{and}$$

$$R(x) = \begin{cases} 1 & \text{if } p|\nu,\\ 1+p^{-3}x & \text{otherwise.}\end{cases}$$

b) if p ramifies in K, then $Z_{\xi,\lambda,p}(x_\pi) = R(x_\pi)/Q(x_\pi)$, where

$$Q(x) = 1 + \frac{(p-\lambda_\pi\overline{\chi}(\pi))}{p^2}x + \frac{1}{p^2}x^2, \qquad \text{and}$$

$$R(x) = \begin{cases} 1 & \text{if } p\in\Sigma,\\ 1+p^{-1}x & \text{otherwise.}\end{cases}$$

c) If p splits in K, and π is a prime of K dividing p, let (α,α') be the solution of the following system of linear equation:

$$\left(\chi(\overline{\pi})\kappa^*(\pi)p^{1/2}\right)\alpha + \left(\chi(\overline{\pi})p^2\right)\alpha' = \lambda_\pi - \chi(\pi)\kappa^*(\overline{\pi})p^{1/2}$$

$$\overline{\left(\chi(\overline{\pi})\kappa^*(\pi)p^{1/2}\right)}\alpha' + \overline{\left(\chi(\overline{\pi})p^2\right)}\alpha = \lambda_{\overline{\pi}} - \overline{\chi(\pi)\kappa^*(\overline{\pi})p^{1/2}}.$$

Then $Z_{\xi,\lambda,p}(x_\pi,x_{\overline{\pi}})$ is the rational function given by

$$Z_{\xi,\lambda,p}(x_\pi,x_{\overline{\pi}}) = \frac{P(x_\pi)P'(x_{\overline{\pi}}) + Cx_\pi x_{\overline{\pi}}}{Q(x_\pi)Q'(x_{\overline{\pi}})} = 1 + \alpha x_\pi + \alpha' x_{\overline{\pi}} + \dots,$$

where the polynomials P,P',Q,Q' and the constant C are given by

$$Q(x) = 1 - \frac{\lambda_{\overline{\pi}}\overline{\chi}(\pi)}{p^2}x + \frac{\lambda_{\pi}\overline{\chi}(\pi)}{p^3}x^2 - \frac{\overline{\chi}^2(\pi)}{p^3}x^3,$$

$$Q'(x) = 1 - \frac{\lambda_{\pi}\overline{\chi}(\overline{\pi})}{p^2}x + \frac{\lambda_{\overline{\pi}}\overline{\chi}(\overline{\pi})}{p^3}x^2 - \frac{\overline{\chi}^2(\overline{\pi})}{p^3}x^3,$$

$$P(x) = 1 - \left(\frac{\overline{\kappa}^*(\pi)\alpha'}{p^{1/2}} + \frac{\overline{\chi}^2\kappa^*(\pi)}{p^{3/2}}\right)x + \frac{\overline{\kappa}^*(\pi)}{p^{5/2}}x^2,$$

$$P'(x) = 1 - \left(\frac{\overline{\kappa}^*(\overline{\pi})\alpha}{p^{1/2}} + \frac{\overline{\chi}^2\kappa^*(\overline{\pi})}{p^{3/2}}\right)x + \frac{\overline{\kappa}^*(\overline{\pi})}{p^{5/2}}x^2, \qquad \textit{and}$$

$$C = \frac{\alpha\chi^2\overline{\kappa}^*(\pi)}{p^{3/2}} + \frac{\alpha'\overline{\chi}^2\kappa^*(\pi)}{p^{3/2}} - \left(\alpha\alpha'\kappa^*(\pi)\overline{\kappa}^*(\pi)p^{-1} + p^{-2}\right),$$

with the convention that $\kappa^*(\pi) = 0$ when $\pi|\nu$.

Proof: We will first show the action of the Hecke operators on modular forms induces an action of the Hecke operators on the formal Dirichlet series Z_θ and that a modular form F is a simultaneous eigenform of the Hecke operators if and only if for each $\xi \in \Xi$ and each $\theta \in V_\xi$, the formal Dirichlet series Z_θ is an eigenfunction for the Hecke operators. Then we will show that the Hecke operators act on the local factors $Z_{\theta,p}$ and, finally, we show that Z_θ is the product of the local factors.

For $\theta \in V_\xi$, let $\{c_\theta(a)\}$ be the coefficients of the formal Dirichlet series Z_θ associated to F, so that

$$F = \sum_{\xi\in\Xi}\sum_{\theta\in\mathcal{B}_\xi}\sum_{a} c_\theta(a)\,(l(a)\theta)\,.$$

Thus, if T_π is a Hecke operator, we see that

$$T_\pi F = \sum_{\xi\in\Xi}\sum_{\theta\in\mathcal{B}_\xi}\sum_{a} c_\theta(a)\,(T_\pi\,(l(a)\theta))\,,$$

and it behooves us to investigate the action of T_π on $l(a)\theta$ as described in §4.6.

Let θ be a primitive Shintani eigenfunction, $\theta \in V_\xi$, and let a be an integral ideal of K; then we can combine Lemmas 4.2(a) and 4.6 to determine how the Hecke operator T_π acts on $l(a)\theta$. There are three cases, depending on the splitting of π.

Consider first the case where $\pi = p$ is an inert prime of K, and let $a = p^j a'$ where a' is relatively prime to p, then by Lemma 4.6(a)

$$T_p l(a)\theta = \chi(p)\,l(p)l(a)\theta + \delta_{p,\nu\mathcal{N}(a)}\,\chi(p)p\,l(a)\theta + \chi(p)p^4\,l(p^{-1})l(a)\theta,$$

where $\delta_{p,n}$ is 1 if $p|n$ and is 0 otherwise. Since p is an inert prime, it is principal in K and so $\chi(p) = 1$. It is also clear by definition of δ that $\delta_{p,\nu\mathcal{N}(a)} = \delta_{p,\nu p^{2j}}$ Thus, applying these observations and using the results of Lemma 4.2(a) (where we take for F the theta function $l(a)\theta \in V_{k,\nu\mathcal{N}(a)}$, we find that $T_p l(p^j a')\theta =$

$$l(p^{j+1}a')\theta + \delta_{p,\nu p^{2j}}\, p\, l(p^j a')\theta + \begin{cases} p^4\, l(p^{j-1}a')\theta & \text{if } j>0, \\ 0 & \text{if } j=0. \end{cases}$$

Notice that this implies that T_p maps the subspace $\bigoplus_j l(p^j a')(\mathbf{C}\theta)$ into itself, and so will induce a relation among the coefficients $\{c_\theta(p^j a') : j \in \mathbf{Z}\}$.

Next, suppose that π is ramified and divides the rational prime p, and let $a = \pi^j a'$ where a' is relatively prime to π. By Lemma 4.6(b), we have

$$T_\pi l(a)\theta = \chi(\pi)\, l(\pi) l(a)\theta + \chi(\pi) p\, l_\pi l(a)\theta + \chi(\pi) p^2\, l(\pi^{-1}) l(a)\theta,$$

and by Lemmas 4.2(a) and 4.4.2, we can simplify this to:

$T_p l(p^j a')\theta =$

$$\chi(\pi)\, l(\pi^{j+1}a')\theta + \chi(\pi) p\, \delta'_{\pi,\Sigma,j}\, l(\pi^j a')\theta + \begin{cases} \chi(\pi) p^2\, l(\pi^{j-1}a')\theta & \text{if } j>0, \\ 0 & \text{if } j=0, \end{cases}$$

where $\delta'_{\pi,\Sigma,j}$ is 1 if π is in Σ or if $j>0$ and 0 otherwise.

Finally, suppose that $\pi \neq \overline{\pi}$ and that π divides the prime p of $\mathbf{Q}$, and let $a = \pi^i \overline{\pi}^j a'$ where a' is relatively prime to p. By Lemma 4.6(c) we see that

$$T_\pi l(a)\theta = \chi(\overline{\pi})\, l(\pi) l(a)\theta + \chi(\pi) p\, l(\overline{\pi}/\pi) l(a)\theta + \chi(\overline{\pi}) p^2\, l(\overline{\pi}^{-1}) l(a)\theta.$$

So by Lemma 4.2(a) $T_\pi l(a)\theta = T_1 l(a)\theta + T_2 l(a)\theta + T_3 l(a)\theta$ where

$$T_1 l(\pi^i \overline{\pi}^j a')\theta = \chi(\overline{\pi}) l(\pi^{i+1}\overline{\pi}^j a')\theta,$$

$$T_2 l(\pi^i \overline{\pi}^j a')\theta = \begin{cases} \chi(\pi) p\, l(\pi^{i-1}\overline{\pi}^{j+1} a')\theta & \text{if } i>0, \\ \chi(\overline{\pi}) p^{1/2} \kappa^*(\pi) l(\overline{\pi}^j a')\theta & \text{if } i=0, \end{cases}$$

$$T_3 l(\pi^i \overline{\pi}^j a')\theta = \begin{cases} \chi(\overline{\pi}) p^2\, l(\pi^i \overline{\pi}^{j-1} a')\theta & \text{if } j>0, \\ \chi(\overline{\pi}) p^{3/2}\, \kappa^*(\pi) l(\pi^{i-1} a')\theta & \text{if } j=0,\ i>0, \\ 0 & \text{if } i=j=0. \end{cases}$$

Observe that we have shown that the Hecke operators induce an action on the formal Dirichlet series Z_θ of a modular form F, and if F is a simultaneous eigenform of the Hecke operators with eigenvalues λ_π then

$$T_\pi Z_\theta = \lambda_\pi Z_\theta \qquad \forall \pi.$$

Conversely, if this relation holds for all $\theta \in V_\xi$ and all $\xi \in \Xi$, then the modular form F is an eigeform of T_π.

We will now show that there is only one (non-zero) formal Dirichlet series $Z_{\xi,\lambda}$, up to a constant, that satisfies these equations for all primes π of K and we will find an explicit formula for this unique solution. In addition, we will show that the constant term of such a solution is nonzero, which will imply that the formal Dirichlet can be normalized by requiring the coefficient $z_\theta(1)$ to be 1. This will in turn imply that the adelic theta coefficients are determined by the eigenvalues and by their primitive components.

We prove that there is a unique solution by considering the action of a single Hecke operator T_π on Z_θ and showing that Z_θ can be written in the form

$$Z_{\xi,\lambda,p}\left(\sum_{\substack{a\in I^+\\ p\nmid a}} z_\theta(a)\right),$$

where p is the rational prime divisible by π, the sum is over all integral ideals relatively prime to p, and $Z_{\xi,\lambda,p}$ is the local Euler factor defined in the statement of the Theorem. An immediate consequence of this result is that there exists uniquely an explicit Euler product decomposition of Z_θ of the form

$$Z_\theta = z_\theta(1)\prod_p Z_{\xi,\lambda,p}.$$

This observation will complete the proof of the theorem.

Thus, we must consider the action of T_π for π dividing the rational prime p in three cases, depending on how p splits in K.

Consider first the case where p is inert in K and for any integral ideal a' which is relatively prime to p, let Z_θ be the power series defined by

$$Z_\theta(x) = \sum_{j=0}^{\infty} c_\theta(p^j a')x^j.$$

To simplify the notation we will drop the subscript and write Z for Z_θ for the remainder of the proof. Using the formula derived above for $T_p l(p^j a')\theta$, we find that

$$T_p\left(\sum_{j=1}^{\infty} c_\theta(p^j a')l(p^j a')\theta\right) = \sum_{j=1}^{\infty} c_\theta(p^j a')T_p l(p^j a')\theta$$

$$= \delta_{p,\nu} p c_\theta(a')l(a')\theta + \sum_{j=1}^{\infty}\left(c_\theta(p^{j-1}a') + pc_\theta(p^j a') + p^4 c_\theta(p^{j+1}a')\right) l(p^j a')\theta,$$

where $\delta_{p,\nu}$ is 1 if $p|\nu$ and is 0 otherwise. Suppose now that $T_p Z = \lambda_p Z$; then we easily verify that

$$\lambda_p Z(x) = xZ(x) + p\left(Z(x) - Z(0)\right) + \delta_{p,\nu} pZ(0) + p^4\left(Z(x) - Z(0)\right)/x.$$

Thus, after some simplification, we find

$$\left(x^2 + (p-\lambda_p)x + p^4)\right) Z(x) = \left(p^4 + \delta_{p,\nu} px\right) Z(0).$$

This shows that

$$Z(x) = c_\theta(a')\,\frac{1+\delta_{p,\nu}p^{-3}x}{1+(p-\lambda_p)p^{-4}x+p^{-4}x^2},$$

as claimed.

Consider now the case where p is ramified in K and let π be the prime of K dividing p. As before let $Z(x)$ be the power series defined by

$$Z(x) = \sum_j c_\theta(\pi^j a')x^j,$$

where a' is an integral ideal relatively prime to π, and suppose that $T_\pi Z_\theta = \lambda_\pi Z_\theta$, then, in the same manner as above, we find that

$$\lambda_\pi Z(x) = \chi(\pi)xZ(x) + \chi(\pi)p\,(Z(x)-Z(0)) + \chi(\pi)p\,\delta'_{\pi,\Sigma}Z(0) + \chi(\pi)p^2\,(Z(x)-Z(0))\,/x,$$

where $\delta'_{\pi,\Sigma}$ is 1 if $\pi\in\Sigma$ and 0 otherwise. Thus,

$$\left(\chi(\pi)x^2 + (\chi(\pi)p-\lambda_\pi)\,x + \chi(\pi)p^2\right)Z(x) = \left(\chi(\pi)p^2 + p\delta'_{\pi,\Sigma}x\right)Z(0),$$

and so $Z(x)$ has the form

$$Z(x) = c_\theta(a')\,\frac{1+\chi(\pi)^{-1}\delta'_{\pi,\Sigma}p^{-1}x}{1+(p-\chi(\pi)^{-1}\lambda_\pi)\,p^{-2}x+p^{-2}x^2},$$

as was to be shown.

Finally, suppose that p splits in K and let π be a prime of K dividing p. As before let a' be an integral ideal of K which is prime to p and let $Z(x,y)$ be the power series defined by

$$Z(x,y) = \sum_{i,j=0}^{\infty} c_\theta(\pi^i\overline{\pi}^j a')x^i y^j,$$

and suppose that $T_\pi Z_\theta = \lambda_\pi Z_\theta$ and $T_{\overline{\pi}}Z_\theta = \lambda_{\overline{\pi}}Z_\theta$.

The action of T_π and $T_{\overline{\pi}}$ on F induces an action on $Z(x,y)$ as described above, and hence provides us with two equations that Z must satisfy. Let us first introduce some notation before solving these equations. Let

$$a=\chi(\overline{\pi})\quad b=\chi(\pi)p\quad c=\chi(\overline{\pi})p^2\quad d=p^{-1/2}\kappa^*(\overline{\pi})\quad \lambda=\lambda_\pi\quad \lambda'=\lambda_{\overline{\pi}},$$

then we see that

$$\lambda Z(x,y) = axZ(x,y) + b(y/x)\,(Z(x,y)-Z(0,y)) + bdZ(0,y)$$

$$+(c/y)\,(Z(x,y)-Z(x,0)) + c\overline{d}(1/x)\,(Z(x,0)-Z(0,0))$$

$$\lambda' Z(x,y) = \overline{a}yZ(x,y) + \overline{b}(x/y)\,(Z(x,y)-Z(x,0)) + \overline{bd}Z(x,0)$$

$$+(\overline{c}/x)(Z(x,y)-Z(0,y))+\overline{c}d(1/y)(Z(0,y)-Z(0,0)).$$

Observe that we need only find $Z(x,0)$ and $Z(0,y)$ and we can then use $Z(x,0)$ and $Z(0,y)$ to solve for $Z(x,y)$ using the first equation. To this end, we set $y=0$ in the previous two equations, to obtain:

$$\lambda Z(x,0)=axZ(x,0)+bdZ(0,y)+cZ_y(x,0)+c\overline{d}(1/x)(Z(x,0)-Z(0,0))$$

$$\lambda' Z(x,0)=\overline{b}xZ_y(x,0)+\overline{bd}Z(x,0)+(\overline{c}/x)(Z(x,0)-Z(0,0))+\overline{c}dZ_y(0,0),$$

where $Z_y(x,y)=\frac{d}{dy}Z(x,y)$ is the derivative of Z with respect to y and so $Z_y(0,0)$ is the coefficient of y in $Z(x,y)$, and we eliminate the term $Z_y(x,0)$ to obtain a formula for $Z(x,0)$:

$$Z(x,0)=\frac{c\overline{c}Z(0,0)-\left(c\overline{c}dZ_y(0,0)+c\overline{bd}Z(0,0)\right)x+b\overline{b}dZ(0,0)x^2}{c\overline{c}-c\lambda' x+\overline{b}\lambda x^2-a\overline{b}x^3}.$$

The term $Z_y(0,0)$ can by determined by setting both x and y to zero in the original pair of equations, which yields

$$\lambda Z(0,0)=bdZ(0,0)+cZ_y(0,0)+c\overline{d}Z_x(0,0)$$

$$\lambda' Z(0,0)=\overline{bd}Z(0,0)+\overline{c}Z_x(0,0)+\overline{c}dZ_y(0,0).$$

If $Z(0,0)$ is nonzero, then these two equations have a unique solution $Z_x(0,0)=\alpha Z(0,0)$, $Z_y(0,0)=\alpha' Z(0,0)$.

Suppose however that $Z(0,0)=0$ then it is easy to see that $Z_y(0,0)$ and $Z_x(0,0)$ are both zero. Indeed, recall that $d=p^{-1/2}\kappa^*(\overline{\pi})$ so if p divides ν, then $d=0$ and we see directly that $Z_x(0,0)$ and $Z_y(0,0)$ are both zero. If p does not divide ν, then $d\overline{d}=p^{-1}$ and the previous two formulas imply that $Z_y(0,0)=d\overline{d}Z_y(0,0)$, so $Z_y(0,0)=0$ and similarly $Z_x(0,0)=0$.

From the observations in the previous paragraph it follows that if $Z(0,0)$ is zero, then $Z_y(0,0)$ is zero, and so $Z(x,0)$ is zero. A symmetric argument shows that $Z(0,y)$ is also zero in this case and hence the original equations have the form

$$\lambda Z(x,y)=axZ(x,y)+b(y/x)Z(x,y)+(c/y)Z(x,y),$$

$$\lambda' Z(x,y)=\overline{a}yZ(x,y)+\overline{b}(x/y)Z(x,y)+(\overline{c}/x)Z(x,y),$$

which implies that $(\lambda xy-ax^2y-by^2-cx)Z(x,y)=0$ and since a,b,c are nonzero, this implies that $Z(x,y)=0$ if $Z(0,0)=0$.

Thus, we can assume henceforth that $Z(0,0)\neq 0$, and we let $\alpha=Z_x(0,0)/Z(0,0)$, $\alpha'=Z_y(0,0)/Z(0,0)$ be the complex numbers defined above. The formula for $Z(x,0)$ can then be written:

$$Z(x,0)=Z(0,0)\,\frac{c\overline{c}-\left(c\overline{c}d\alpha'+c\overline{bd}\right)x+b\overline{b}dx^2}{c\overline{c}-c\lambda' x+\overline{b}\lambda x^2-a\overline{b}x^3},$$

and by a symmetic argument we find that

$$Z(0,y)=Z(0,0)\,\frac{c\overline{c}-\left(c\overline{c}\overline{d}\alpha+\overline{c}bd\right)y+b\overline{b}\overline{d}y^2}{c\overline{c}-\overline{c}\lambda y+b\lambda' y^2-\overline{a}by^3}.$$

Thus, if we substitute these formulas into the original equations, we have

$$Z(x,y) = \frac{\left(c\bar{d}y - cx\right) Z(x,0) + \left(bdxy - by^2\right) Z(0,y) - c\bar{d}y}{R(x,y)},$$

where $R(x,y) = \lambda xy - ax^2y - by^2 - cx$. If we let $P(x)$ be the numerator of $Z(x,0)$ and $Q(x)$ the denominator, (both normalized to have constant term 1), and similarly define $P'(y)$, $Q'(y)$ for $Z(0,y)$, then we find that

$$Z(x,y) = c_\theta(a') \, \frac{\left(c\bar{d}y - cx\right) P(x)Q'(y) + \left(bdxy - by^2\right) P'(y)Q(x) - c\bar{d}yQ(x)Q'(y)}{Q(x)Q'(y)R(x,y)}.$$

One can then check (using a symbolic manipulation program if available) that the numerator of this equation is divisible by $R(x,y)$ and that the quotient is $P(x)P'(y) + Cxy$ where C is the constant given in the statement of the theorem.

We have now shown that if p is any rational prime, then

$$Z_\theta = Z_{\xi,\lambda,p} \left(\sum_a{}' c_\theta(a) \right),$$

where the sum is over all integral ideals that are relatively prime to p and $Z_{\xi,\lambda,p}$ is the Euler factor given in the theorem. From this we see that F is a simultaneous eigenform of the Hecke operators with eigenvalues $\lambda = \{\lambda_\pi\}$ if and only if

$$Z_\theta = c_\theta(1) \prod_p Z_{\xi,\lambda,p},$$

for all $\theta \in V_\xi$ and all $\xi \in \Xi$. **Q.E.D.**

CHAPTER 5

EULER PRODUCTS I: THE SIEGEL FUNCTION

In this Chapter, we prove that each of the completions G_p admit Iwasawa decompositions $G_p = P_p G_p(L_p)$ with respect to the lattice $L_p = O_p^3$ (Lemma 5.4.3,5.5.2) and we explicitly evaluate the Siegel function S_Λ which occurs in the Siegel-Baily-Tsao-Karel integral (cf. Lemma 2.7) by showing that it can be written as a product of local factors and explicitly evaluating these local factors (Propositions 5.4.5,5.4.6,5.5.5).

Recall that the Siegel function, S_Λ, is the continuous function on $M_{\mathbf{A}} \times \mathbf{Q_A}$ defined by

$$S_\Lambda(m,r) = \int_{N_{\mathbf{A}}} \Lambda(\delta(\iota n m))\, \lambda(-r\cdot n)\, dn,$$

where ι is a representative of the non-trivial element of the Weyl group (§1.2), λ is the unique continuous character of $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}/N(L)_f$ such that $\lambda([0,t_\infty]) = e(t_\infty)$(§2.4), δ is the Iwasawa function (§1.5),

$$\delta : G_{\mathbf{A}}/G(L)_f \mathbf{K}^0_\infty \to D_{\mathbf{A}}/D(L)_f Z_1, \qquad g \in \delta(g) U_{\mathbf{A}} G(L)_f \mathbf{K}^0_\infty,$$

and Λ is a Hecke character of weight k on the torus $D_{\mathbf{A}}$ (§1.7), that is, Λ is a continous homomorphism from the group $D_{\mathbf{Q}}\backslash D_{\mathbf{A}}/D(L)_f$ into $\mathbf{C}^*$ such that

$$\Lambda(d_\infty d_f) = jac(d_\infty, o)^k \Lambda(d_f),$$

for all $d = d_\infty d_f \in D_{\mathbf{A}}$. As a function of m, $S_\Lambda(m,r)$ is invariant under right translation by the subgroup $Z_\infty M(L)_f$.

The results in this chapter show that for each fixed $m \in M_{\mathbf{A}}$ the Siegel function $S_\Lambda(m,r)$ admits an Euler product. More precisely, if $m = a[w,t]$ (c.f. 1.2), then the Siegel function can be written

$$S_\Lambda(a[w,t],r) = \frac{(-2\pi i)^{3k}}{(3k-1)!}\, r_\infty^{3k-1}\, j_k(m_\infty, r_\infty) \prod_p S_{\Lambda,p}(a_p[w_p,t_p],r_p),$$

where $j_k(m_\infty, \nu\|a_f\|) = j_{k,\nu}(au)$ is the factor of automorphy for adelic theta functions of level ν and weight k and the local factors $S_{\Lambda,p}(a_p[w_p,t_p],r_p)$ are finite polynomials in $p^{-3k/2}$ and certain values of Λ which depend on a_p and t_p in a simple way:

$$S_{\Lambda,p}(a_p[w_p,t_p],r_p) = \lambda(r_p a_p \overline{a}_p t_p)\, \Lambda(\overline{a}_p^{-1})\, \|a_p\|\, S_{\Lambda,p}([w_p,0], r_p a_p \overline{a}_p).$$

Moreover, if r_p is a unit in $\mathbf{Z}_p$ and $a_p = 1$,then the value of the local Siegel function has a relatively simple form which is most conveniently given in three cases depending on whether p remains inert, ramifies, or splits in K. If p remains inert in K then

$$S_{\Lambda,p}([w_p,t_p],r_p) = \begin{cases} \lambda_p(r_p(t_p + w_p\overline{w}_p D/2)) \ \left(1-p^{-3k}\right) & \text{if } w_p \in O_p, \\ 0 & \text{otherwise.} \end{cases}$$

If p ramifies in K and π is the prime of K dividing p, then

$$S_{\Lambda,p}([w_p,t_p],r_p) = \begin{cases} \lambda_p(r_p(t_p + w_p\overline{w}_p D/4)) \ \left(1-p^{-3k}\right) & \text{if } w_p \in O_p, \\ \lambda_p(r_p(t_p + w_p\overline{w}_p D/4)) \ \left(\Lambda_T^*(\pi)p^{-3k/2} - p^{-3k}\right) & \text{if } w \in \pi^{-1}O_p^*, \\ 0 & \text{otherwise,} \end{cases}$$

where Λ_T is the restriction of Λ to T and $\Lambda_T^*(x) = \Lambda_T(x)\|x\|^{-3k/2}$ is the associated unitary Hecke character. In the case of p ramified, $\Lambda_T^*(\pi) = \pm 1$.

Finally, if p splits in K, then K_p is a ring, and there are idempotents e, $\overline{e}$ of K such that $e+\overline{e} = 1$, $e\overline{e} = 0$. If we let $\pi = pe+\overline{e}$, then π is a prime of O_p and $p = \pi\overline{\pi}$ and the generator $\tau = \sqrt{-D}$ can be expressed as $\tau = \tau_1 e - \tau_1\overline{e}$ for some $\tau_1 \in \mathbf{Q}_p$. The values $S_{\Lambda,p}([w_p,t_p],r_p)$ are most easily described with respect to the partition $C_{a,b} = \pi^{-a}\overline{\pi}^{-b}O_p^* + O_p$. Indeed,

$$S_{\Lambda,p}([w_p,t_p],r_p) = \begin{cases} \lambda_p(r_p(t_p + w_p\overline{w}_p\tau_1/2)) \ \left(1-p^{-3k}\right) & \text{if } w_p \in O_p, \\ \lambda_p(r_p(t_p + w_p\overline{w}_p\tau_1/2))\,\overline{\chi}_1(\pi)^a \left(1-\chi_1(\pi)\right) & \text{if } w_p \in C_{a,0}, \\ \lambda_p(r_p(t_p - w_p\overline{w}_p\tau_1/2)\,\overline{\chi}_1(\overline{\pi})^b \left(1-\chi_1(\overline{\pi})\right) & \text{if } w_p \in C_{0,b}, \\ 0 & \text{if } w_p \in C_{a,b},\ a,b \geq 1, \end{cases}$$

where $\chi_1(x) = \|x\|^{3k/2}\,\Lambda_Z^2\Lambda_T^{*3}(x)$, and Λ_Z, Λ_T are the restrictions of Λ to the subtori Z, T of D (c.f. §1.7).

The bulk of the calculation is involved in showing that G_p admits an Iwasawa decomposition, $G_p = D_pU_pG(L)_p$, and in explicitly evaluating the local Iwasawa function (§cf. 1.5)

$$\delta_p : G_p/G(L)_p \to D_p/D(L)_p,$$

defined by $g_p \in \delta_p(g_p)U_pG(L)_p$. In §5.1, we show that S_Λ admits a factorization indexed by the valuations of $\mathbf{Q}$. In the remaining sections we explicitly evaluate the factors corresponding to the infinite prime, the inert primes, the ramified primes, and the split primes, respectively. The most difficult case is that when p splits.

5.1. Local Siegel functions

Since $\Lambda \circ \delta$, and λ are defined as products of local factors, the Siegel function admits a factorization

$$S_\Lambda(m,r) = S_{\Lambda,\infty}(m_\infty,r_\infty)\prod_p S_{\Lambda,p}(m_p,r_p),$$

where the archimedean factor is

$$S_{\Lambda,\infty}(m_\infty,r_\infty) = \int_{N_\infty} \Lambda(\delta(\iota_\infty n_\infty m_\infty))\lambda_\infty\left(-r_\infty \cdot n_\infty\right) dn_\infty,$$

and the nonarchimedean factors are

$$S_{\Lambda,p}(m_p,r_p)=\int_{N_p}\Lambda(\delta(\iota_p n_p m_p))\lambda_p\left(-r_p\cdot n_p\right)dn_p.$$

We will denote the product of all the nonarchimedean factors by $S_{\Lambda,f}(m_f,r_f)$, so that $S_\Lambda(m,r)=S_{\Lambda,\infty}(m_\infty,r_\infty)S_{\Lambda,f}(m_f,r_f)$

In the following lemma, we show that the Siegel function behaves nicely with respect to left translation by elements of $T\cdot N$; but it is not left-$M_{\mathbf{Q}}$ invariant.

Lemma 5.1. *Let $m\in M_{\mathbf{A}}$, $n\in N_{\mathbf{A}}$, $x\in T_{\mathbf{A}}$, and $r\in\mathbf{Q}_{\mathbf{A}}$, then*

$$S_\Lambda(nxm,r)=\lambda(r\cdot n)\Lambda(\overline{x}^{-1})\|x\|S_\Lambda(m,x\overline{x}r),$$

and $\Lambda(\overline{x}^{-1})=\Lambda(x)\|x\|^{-3k}$.

Proof: First observe that the translation invariance of Haar measure for $N_{\mathbf{A}}$ implies that

$$S_\Lambda(nxm,r)=\lambda(r\cdot n)S_\Lambda(xm,r).$$

So it will suffice to show that

$$S_\Lambda(xm,r)=\Lambda(\overline{x}^{-1})\|x\|S_\Lambda(m,x\overline{x}r).$$

To prove this assertion, first recall that by construction (§1.5) δ restricted to $D_f/D(L)_f$ is the identity map. Thus, since Λ is a homomorphism, we have

$$\Lambda\delta(\iota nxg)=\Lambda\delta(\iota xx^{-1}nxg)=\Lambda\delta(\overline{x}^{-1}\iota x^{-1}nxg)=\Lambda(\overline{x}^{-1})\Lambda\delta(\iota x^{-1}nxg),$$

for any $x\in T_f$, $n\in N_{\mathbf{A}}$, and $g\in G_f$. Thus, the integral in question can be written

$$S_\Lambda(xm,r)=\int_{N_{\mathbf{A}}}\Lambda(\delta(\iota nxm))\ \lambda\left(-r\cdot n\right)\ dn$$

$$=\Lambda(\overline{x}^{-1})\int_{N_{\mathbf{A}}}\Lambda(\delta(\iota x^{-1}nxm))\lambda\left(-r\cdot n\right)\ dn.$$

If we make the change of coordinates $n_1=x^{-1}nx$, and observe that $n_1=(x\overline{x})^{-1}\cdot n$ and $dn_1=\|x\|^{-1}dn$, we obtain

$$S_\Lambda(xm,r)=\Lambda(\overline{x}^{-1})\|x\|\int_{N_{\mathbf{A}}}\Lambda(\delta(\iota n_1 m))\lambda\left(-rx\overline{x}\cdot n_1\right)\ dn_1.$$

To complete the proof of the lemma we must show that $\Lambda(\overline{x}^{-1})=\Lambda(x)\|x\|^{-3k}$ and to show this it will suffice to show that if $a\in\mathbf{Q}_{\mathbf{A}}^*$ then $\Lambda(a)=\|a\|^{3k/2}$ but this follows easily from the fact (§1.7) that $\Lambda(x)/\|x\|^{3k/2}$ is a unitary Hecke character and so is identically 1 on $\mathbf{Q}_{\mathbf{A}}^*$. **Q.E.D.**

This lemma implies that if $m = a[w,t] \in T_{\mathbf{A}}U_{\mathbf{A}}$, then

$$S_\Lambda(a[w,t],r) = \lambda(rt)\,\Lambda(a)\,\|a\|^{-(3k-1)}\,S_\Lambda([w,0],ra\overline{a}),$$

and so the only interesting values of $S_\Lambda(m,r)$ are the values at elements of the form $m = [w,0]$.

5.2. Evaluation of the archimedean factor

In this section we show that the archimedean factor of the Siegel function is essentially the factor of automorphy for adelic theta functions. Before stating the Proposition we recall the definition of this factor of automorphy (§3.1). First, recall that we have defined complex valued functions ξ_1, ξ_2 on M_∞ by

$$m_\infty \cdot o = (\xi_1(m_\infty), \xi_2(m_\infty), 1),$$

where $o = (\sqrt{-D}/2, 0, 1)$ is the base point in the definition of adelic modular forms. The values of these functions on elements of the form $m_\infty = [w,t]a$ are easily seen to be

$$\xi_1(m_\infty) = t + (\|a\| + \|w\|)\sqrt{-D}/2, \qquad \xi_2(m_\infty) = w.$$

The factor of automorphy for adelic theta functions is then given on $m = ua$ by

$$j_{k,\nu}(m) = j_k\left(m, \nu/\mathcal{N}(a_f)\right),$$

where j_k is the function on $M_\infty \times \mathbf{R}$ defined by

$$j_k(m,r) = jac(m,o)^k e(r\,\xi_1(m)).$$

Proposition 5.2. *Notation as above, let $m \in M_\infty$ and $r \in \mathbf{R}$. Then*

$$S_{\Lambda,\infty}(m,r) = c_1 r^{3k-1} j_k(m,r),$$

where $c_1 = (-2\pi i)^{3k}/(3k-1)!$.

Proof: Since $\Lambda(\delta(g_\infty)) = jac(g_\infty, o)^k$ (cf. §1.7) we find that

$$S_{\Lambda,\infty}(m,r) = \int_{N_\infty} jac(\iota n m, o)^k e\left(-rn\right) dn.$$

The cocycle relation satisfied by jac allows us to write

$$jac(\iota n m, o) = jac(\iota, nmo) jac(m,o),$$

since $jac(n,\xi) = 1$ for all ξ. Furthermore, if we use the explicit formulas of §§1.2-3 to evaluate the jacobian determinant of ι at the point $m \cdot o$, we find that

$$jac(\iota, m \cdot o) = \xi_1(m)^{-3},$$

and since $\xi_1([0,s]m) = s + \xi_1(m)$, we find that

$$S_{\Lambda,\infty}(m,r) = jac(m,o)^k \int_{-\infty}^{\infty} (s+\xi_1(m))^{-3k} e^{-2\pi i r s} ds.$$

Since $Im\xi_1(m) > 0$ and $r > 0$, the integrand has a pole in the lower half plane and the integrand vanishes rapidly on the semicircle of radius R in the lower half plane as R gets large. Thus, we may use the residue theorem to evaluate this integral. The residue of $(t+\xi)^{-3k}e^{ct}$ at $-\xi$ is easily seen to be $c^{3k-1}e(-c\xi)/(3k-1)!$. Thus, if γ is a path encircling the residue in a counterclockwise direction, we have

$$c^{3k-1}e(-c\xi)/(3k-1)! = \frac{-1}{2\pi i}\int_{\gamma}(s+\xi_1(m))^{-3k}e^{-2\pi i r s} ds =$$

$$\frac{-1}{2\pi i}\int_{-\infty}^{\infty}(s+\xi_1(m))^{-3k}e^{-2\pi i r s} ds,$$

and hence we have evaluated the archimedean component of the Siegel function; we have

$$S_{\Lambda,\infty}(m,r) = jac(m_\infty,o)^k(-2\pi i)^{3k}\, r^{3k-1}\; e\left(r\xi_1(m)\right)/(3k-1)! \;\;,$$

which is just another way of writing the formula in the Proposition. **Q.E.D.**

5.3. Exponential Sums

The evaluation of the nonarchimedean factors of the Siegel function will involve certain exponential sums. In this section, we introduce some notation to deal uniformly with these sums.

Let p be a rational prime, let $x, y \in \mathbf{Q}_p^*$, and let $\beta_x(y)$ and $\beta_x^*(y)$ be the exponential sums defined by

$$\beta_x(y) = \int_{x\mathbf{Z}_p} \lambda_p(yt)dt, \qquad \beta_x^*(y) = \int_{x\mathbf{Z}_p^*} \lambda_p(yt)dt,$$

where λ_p is the unique character of $\mathbf{Q}_p/\mathbf{Z}_p$ such that $\lambda_p(p^{-s}) = e(-1/p^s)$ for all $s \geq 0$. Then it is easy to see that

$$\beta_x(y) = \|x\|\beta_1(xy), \qquad \beta_x^*(y) = \|x\|\beta_1(xy), \qquad \beta_1^*(x) = \beta_1(x) - p^{-1}\beta_1(px),$$

and that β_1 is the characteristic function of $\mathbf{Z}_p$. Thus, for $r \in p^c\mathbf{Z}_p^*$,

$$\beta_1(r) = \begin{cases} 1 & \text{if } c \geq 0, \\ 0 & \text{if } c < 0; \end{cases} \qquad \beta_1^*(r) = \begin{cases} 1-p^{-1} & \text{if } c \geq 0, \\ -p^{-1} & \text{if } c = -1, \\ 0 & \text{if } c < -1. \end{cases}$$

5.4. Evaluation of the local Siegel function at inert and ramified primes

In this section, we assume that p is a rational prime that either ramifies in K or is inert in K. Under these assumptions we prove that G_p admits an Iwasawa decomposition with respect to the compact subgroup $G(L)_p$ with $L = O_p^3$, and we explicitly evaluate the local Iwasawa function δ_p on elements in the set ιU_p (5.4.4). We then use these results to explicitly evaluate the local Siegel function (Prop. 5.4.5, 5.4.6).

5.4.1. Preliminaries

Since, by assumption, p does not split in K, the completion $K_p = K \otimes \mathbf{Q}_p$ is a quadratic field extension of $\mathbf{Q}_p$ and $\tau = \sqrt{-D}$ is the generator of K_p over $\mathbf{Q}_p$. A few simple calculations shows that the ring of integers of K_p can be expressed as $O_p = \mathbf{Z}_p + \mathbf{Z}_p\omega_p$ where ω_p is defined by

$$\omega_p = \begin{cases} (D+\tau)/2 & \text{if } p \text{ is inert in } K, \\ (D/2+\tau)/2 & \text{if } p \text{ ramifies in } K. \end{cases}$$

If p is ramified in K, then ω_p generates the prime ideal of O_p. If p is inert then ω_p is a unit of O_p and, by definition, p generates the prime ideal of O_p. From these observations, it follows that the compact subgroup $U(L)_p$ consists of the elements

$\{[w, t + w\overline{w}D/2] : w \in O_p,\ t \in \mathbf{Z}_p\}$ if p remains inert;

$\{[w, t + w\overline{w}D/4] : w \in O_p,\ t \in \mathbf{Z}_p\}$ if p ramifies.

Since K_p is a quadratic field extension of $\mathbf{Q}_p$ in the cases considered here, the Iwasawa decomposition for G_p with respect to $G(L)_p$ (5.4.3) will be a consequence of the existence of a Witt decomposition for maximal lattices in K_p^3. Moreover, the proof of the Iwasawa decomposition will show us how to explicitly calculate the local Iwasawa function δ_p on elements in the set ιU_p, where the δ_p is defined by

$$\delta_p : G_p/G(L)_p \to D_p/D(L)_p, \qquad g_p \in \delta(g_p)U_pG(L)_p.$$

Recall that the torus D_p is the product of two tori Z_p, T_p each isomorphic to K_p^*, via the map $d(z,t) = diag(z\overline{t}, z, zt^{-1})$. In Proposition 5.4.4, we show that if π is the prime of K dividing p, and $u = [w,t] \in U_p$, then

$$\delta_p(\iota u) = d(1, \pi^{\alpha(u)}),$$

where

$$\alpha([w,t]) = max\left(0, -ord_\pi(w), -ord_\pi(t - w\overline{w}\tau/2)\right).$$

From this explicit calculation of the local Iwasawa function we will be able to deduce that the function in the case of p inert,

$$f_w(t) = \begin{cases} \delta_p(\iota[w, t - w\overline{w}D/2]) & \text{if } p \text{ is inert in } K, \\ \delta_p(\iota[w, t - w\overline{w}D/4]) & \text{if } p \text{ ramifies in } K, \end{cases}$$

is constant, as a function of t, on the sets $p^{-j}\mathbf{Z}_p^*$. Indeed, its value will be explicitly computed and will be used to explicitly evaluate the local factor $S_{\Lambda,p}$ of the Siegel function. The general formulas for the values of $S_{\Lambda,p}$ are proved in Propositions 5.4.5 and 5.4.6.

5.4.2. Witt decompositions and maximal lattices

We begin with a review of Witt decompositions. Let V be a 3-dimensional K_p vector space and let $\Phi : V \times V \to K_p$ be a nondegenerate, indefinite hermitian form on V with respect to the non-trivial galois automorphism of K_p over $\mathbf{Q}_p$. A Witt decomposition of (V, Φ) is a basis (x, y, z) of V such that the matrix representation of Φ with respect to this basis is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

for some $a \in \mathbf{Q}_p^*$.

The existence of an Iwasawa decomposition for G_p with respect to $G(L)_p$ follows from the fact that $L = O_p^3$ is a maximal lattice in the sense of Shimura [10]. We now describe Shimura's classification of maximal lattices for G_p.

Let V, Φ be as above and let L be a O_p-lattice in V. The norm, $\mu(L)$, of L is the $\mathbf{Z}_p$–ideal generated by the norms of the elements in L with respect to Φ: $\mu(L) = \{\Phi(\lambda, \lambda) : \lambda \in L\}O_p$. An O_p-lattice is maximal if it is maximal among all O_p-lattices of the same norm. Recall that $\tau = \sqrt{-D}$ generates the relative different of K_p over $\mathbf{Q}_p$.

Proposition 5.4.2. *[Shimura, 10, Prop.4.7.]*

(i). *Let L be a maximal O_p-lattice in V, and let x be an isotropic element of V, then there is a Witt decomposition $(\alpha x, y, z)$ of V such that $L = O_p \alpha x + O_p y + \mu(L)\tau^{-1} O_p z$, where $\alpha O_p = \{a \in K_p : ax \in L\}$.*

(ii). *Conversely, let (x, y, z) be a Witt decomposition of V, A a fractional $\mathbf{Z}_p$–ideal, and $B = \{\beta \in K_p : \beta\overline{\beta} \in A\Phi(y, y)^{-1}\}$, then $O_p x + By + O_p A \tau^{-1} z$ is a maximal O_p–lattice of norm A.*

Proof: We refer the reader to [10, Prop. 4.7] for a simple proof of this Proposition. There are a few minor differences between Shimura's Proposition and ours. First of all, we explicitly give ourselves an isotropic x and Shimura only asserts that such an x exists, but his proof makes it clear that any isotropic x would do. Second, in place of $O_p y$ he has an O_p–lattice M in $K_p y$, but by replacing y by a suitable scalar multiple, we may assume that $M = O_p y$. **Q.E.D.**

5.4.3. The Iwasawa decomposition of G_p with respect to $G(L)_p$

We can now deduce the existence of the local Iwasawa decomposition for G_p with respect to $G(L)_p$ from the existence of Witt decompositions for maximal lattices.

Lemma 5.4.3. *Let p be a prime that either remains inert or ramifies in K and let $L_p = O_p^3$, then $G_p = P_p G_p(L_p)$.*

Proof: We will apply the previous proposition in the case where $V = K_p^3$, $\Phi(v, w) = {}^t\overline{v} R w$, and R is the hermitian form given in §1.1:

$$R = \begin{pmatrix} 0 & 0 & \tau \\ 0 & -D & 0 \\ -\tau & 0 & 0 \end{pmatrix},$$

where $\tau = \sqrt{-D}$ generates K_p over $\mathbf{Q}_p$. Let (e_1, e_2, e_3) be the standard basis of K_p^3, then $(e_1, e_2, e_3/\tau)$ is a Witt decomposition of V with respect to Φ, and by the proposition, $O_p^3 = O_p e_1 + O_p e_2 + D\tau^{-1} O_p (e_3/\tau)$ is a maximal O_p-lattice of norm D. Recall that the closure of $G_{\mathbf{Q}}$ in $GL_3(K_p)$ is

$$G_p = \{ g \in GL_3(K_p) : {}^t\overline{g} R g = \mu(g) R,\ \mu(g) \in \mathbf{Q}_p^* \},$$

and that P_p is the subgroup of all h such that $he_1 \in K_p^* e_1$. Moreover, $G(L)_p$ is the subgroup stabilizing the lattice $L_p = O_p^3$.

Let $g \in G_p$ and observe that $L' = gL$ is a maximal O_p-lattice of norm $\mu(g)\mu(L)$. By multiplying g on the right by an element z^{-1} of the center Z_p of G_p such that $\mu(z) = \mu(g)$, we may assume that $\mu(g) = 1$. Indeed, let $z = d(\zeta, 1)$, where $\zeta = det(g)\mu(g)^{-1}$, then $z \in Z_p \subset P_p$ and $\mu(z) = det(g)\overline{det(g)}\mu(g)^{-2} = \mu(g)$, since $det(g)\overline{det(g)} = \mu(g)^3$. Thus, if we let $g_1 = z^{-1}g$, then $\mu(g_1) = 1$.

Now, define an element x' in K_p^3 by $x' = \alpha_g e_1$, where α_g is a generator of the O_p-ideal $\alpha_g O_p = \{ a \in K_p : ae_1 \in g_1 L \}$. By Proposition 5.4.2, there is a Witt decomposition (x', y', z') such that $g_1 L = O_p x' + O_p y' + \mu(L)\tau^{-1} O_p z'$. Let ρ be the K_p linear map which takes the basis $(e_1, e_2, (e_3/\tau))$ to (x', y', z'). Then ρ is in G_p and $\rho e_1 = \alpha e_1$, so ρ is in the parabolic subgroup P_p. Moreover, the element $\lambda = \rho^{-1} g_1$ stabilizes the lattice L_p, so $\lambda \in G(L)_p$, and we have shown that $g = z\rho\lambda \in P_p G(L)_p$. **Q.E.D.**

5.4.4. Evaluation of the Iwasawa function

Let $\delta_p : G_p/G(L)_p \to D_p/D(L)_p$ be the local Iwasawa function, which is defined by the relation: $g \in \delta_p(g) U_p G(L)_p$. Further, let δ_Z and δ_T be the projections of δ_p into the subtori Z and T of D, that is $\delta(g) = d(\delta_Z(g), \delta_T(g))$ where $d : {K_{\mathbf{A}}^*}^2 \to D$ is defined in §1.2.

Lemma 5.4.4. *Let p be a prime that either remains inert or ramifies in K, let π be a generator of the prime ideal of O_p, and let $\tau = \sqrt{-D}$ be the generator of K_p over $\mathbf{Q}_p$. We have the following*

a) *Let $g \in G_p$, then $\delta_Z(g) = det(g)\mu(g)^{-1}$ and if $\delta_Z(g) = 1$, then*

$$\delta_T(g) O_p = \{ a \in K_p : a(g^{-1}e_1) \in L \};$$

b) *For every $u = [w, t] \in U_p$, the local Iwasawa function on ιu is an element of the subtorus $T_p/T(L)_p$ given by*

$$\delta_p(\iota u) = d(1, \pi^{\alpha(u)}),$$

where the exponent $\alpha(u)$ is

$$\alpha([w, t]) = max\,(0, -ord_\pi(w), -ord_\pi(t - w\overline{w}\tau/2)).$$

Proof: In the proof of the Lemma 5.4.3, we showed that if $g \in G_p$, then we could write $g = z\rho\lambda$ where $z \in Z_p$, $\rho \in P_p$, $\lambda \in G(L)_p$, and $\mu(\rho\lambda) = 1$. Moreover, ρ is constructed so that $\rho e_1 = \alpha e_1$ where

$$\alpha O_p = \{ a \in K_p : a(g^{-1}e_1) \in L \}.$$

Since $\lambda L = L$, we must have $\mu(\lambda)\mu(L) = \mu(L)$, and so $\mu(\lambda) \in O_p^*$. It is also clear that $det(\lambda) \in O_p^*$ (since det defines a continuous homomorphism of the compact group $G(L)_p$ into K_p^*, the image must lie in the maximal compact subgroup O_p^* of K_p^*). Since $\mu(\rho\lambda) = 1$, we must have $\mu(\rho) \in O_p^*$.

Now we may assume that $\rho = d(z,x)u$ for some $z, x \in K_p^*$, and some $u \in U_p$. A simple calculation shows that

$$\mu(\rho) = z\overline{z}.$$

Thus, if $z = x\pi^a$ for some $x \in O_p^*$ and $a \in \mathbf{Z}$, then $\mu(\rho) = a\overline{a}\pi^a\overline{\pi}^a$. By assumption, there is some unit $u_0 \in O_p^*$ such that $\overline{\pi} = u_0\pi$, and so

$$\mu(\rho) = a\overline{a}u_0\pi^{2a} \in O_p^*\pi^{2a}.$$

This implies that $a = 0$ and hence $\rho \in T_pU_p$ and so $z = \delta_Z(g)$.

Now, assume that $\delta_Z(g) = 1$, then $g = \rho\lambda$ and $\rho = d(1,x)u$ for some $u \in U_p$ and for some $x \in \delta_T(g)O_p^*$. Since $d(1,x)ue_1 = \overline{x}e_1$, we see that $\delta_T(g) = \alpha$ as claimed. This completes the proof of (a).

By the remarks above, the projection $\delta_Z(\iota u)$ of $\delta_p(\iota u)$ into the subtorus Z is 1, since $det(\iota[w,t])$ and $\mu(\iota[w,t])$ are both 1. Thus, the Lemma follows from the observation that $\delta_T(\iota u)O_p = \{a \in K_p : a((\iota u)^{-1}e_1) \in L\}$ where

$$\left(\iota[w,t]\right)^{-1} e_1 = {}^t(1, -w, -t + w\overline{w}\tau/2).$$

Q.E.D.

5.4.5. Evaluation of the local Siegel function when p is inert

We can now use the explicit formula for the local Iwasawa function to evaluate the local Siegel function $S_{\Lambda,p}(m,r)$ which is defined for $m \in M_p$, $r \in \mathbf{Q}_p^*$ by the integral

$$S_{\Lambda,p}(m,r) = \int_{N_p} \Lambda(\delta(\iota n m))\lambda(-r\cdot n)dn.$$

Proposition 5.4.5. *Let p be a prime which is inert in K, let $q = p^{-(3k-1)}$, and let $\beta_x(y)$ and $\beta_x^*(y)$ be the exponential sums defined in §5.3. Then for every $0 \neq r \in \mathbf{Z}_p$, $x \in T_p$, $[w,t] \in U_p$ we have*

$$S_{\Lambda,p}(x[w,t],r) = \lambda(rx\overline{x}t)\Lambda_T(\overline{x}^{-1})\|x\|S_\Lambda([w,0],x\overline{x}r).$$

Moreover, let $u = [w, -w\overline{w}D/2] \in U_p$ and let $a = max(0, -ord_p(w))$; then we have

$$S_{\Lambda,p}(u,r) = q^{2a}\,\beta_1(r/p^{2a}) \;+\; \sum_{j=2a+1}^{\infty} q^j\,\beta_1^*(r/p^j).$$

This sum is finite, and is zero unless $ord_p(r) \geq 2a$. In particular, if $r \in \mathbf{Z}_p^*$, then

$$S_{\Lambda,p}([w,-w\overline{w}D/2],r) = \begin{cases} 1 - p^{-3k} & \textit{if } w \in O_p, \\ 0 & \textit{otherwise.}\end{cases}$$

Proof: The first assertion, concerning the value of the Siegel function at $x[w,t]$, follows directly from Lemma 5.1, and shows that we need only evaluate the Siegel function on elements of the form $u = [w, t_w]$ where we are free to choose t_w as we please. So we may assume that $u = [w, -w\overline{w}D/2]$ and we must evaluate

$$S_{\Lambda,p}(u,r) = \int_{\mathbf{Q}_p} \Lambda(\delta(\iota[w, t - w\overline{w}D/2]))\lambda_p(-rt)dt.$$

By Proposition 5.4.4, $\delta(\iota u) = d(1, p^{\alpha(u)})$, where

$$\alpha([w, t - w\overline{w}D/2]) = max\left(0, -ord_p(w), -ord_p(t - w\overline{w}\omega_p)\right), \qquad \omega_p = (D+\tau)/2.$$

Since $O_p = \mathbf{Z}_p + \mathbf{Z}_p\omega_p$, we see that $p^jO_p = p^j\mathbf{Z}_p + p^j\mathbf{Z}_p\omega_p$, and hence that

$$\alpha([w, t - w\overline{w}D/2]) = max(2a, -ord_p(t)).$$

Thus we can evaluate the integral over the disjoint open sets on which $\alpha(u)$ is constant, that is over the open sets in the following partition of $\mathbf{Q}_p$

$$\mathbf{Q}_p = p^{-2a}\mathbf{Z}_p \cup \bigcup_{j=2a+1}^{\infty} p^{-j}\mathbf{Z}_p^*.$$

Using this partition we easily see that

$$S_{\Lambda,p}([w, -w\overline{w}D/2], r) \int_{p^{-2a}\mathbf{Z}_p} \Lambda_{T,p}(p^{2a})\lambda_p(-rt)dt$$

$$+ \sum_{j=2a+1}^{\infty} \int_{p^{-j}\mathbf{Z}_p^*} \Lambda_{T,p}(p^j)\lambda_p(-rt)dt$$

$$= \Lambda_{T,p}(p^{2a})\beta_{p^{-2a}}(r) + \sum_{j=2a+1}^{\infty} \Lambda_{T,p}(p^j)\beta^*_{p^{-j}}(r),$$

which reduces to the formula in the Proposition, upon observing (§1.7) that $\Lambda_{T,p}(p) = \Lambda_{T,\infty}(p^{-1}) = p^{-3k}$. **Q.E.D.**

5.4.6. Evaluation of the local Siegel function when p ramifies

Now, let p be a prime that ramifies in K and let $\omega_p = (D/2+\tau)/2$ where $\tau = \sqrt{-D}$ generates K_p over $\mathbf{Q}_p$. Then ω_p generates the prime ideal π of O_p and also generates O_p over $\mathbf{Z}_p$, that is $O_p = \mathbf{Z}_p + \mathbf{Z}_p\omega_p$. From these observations it follows that the upper right entry in the matrix $u = [w, t + w\overline{w}D/4]$ is $t + w\overline{w}\omega_p$ and so

$$U(L)_p = \{u = [w, t + w\overline{w}D/4] : w \in O_p, t \in \mathbf{Z}_p\}.$$

Proposition 5.4.6. *Let p be a prime which ramifies in K, let $q = p^{-(3k-1)}$, let π be a prime of K dividing p, and let $\beta_x(y)$ and $\beta_x^*(y)$ be the exponential sums defined in §5.3. Then for every $0 \neq r \in \mathbf{Z}_p$, $x \in T_p$, $[w,t] \in U_p$ we have*

$$S_{\Lambda,p}(x[w,t],r) = \lambda(rx\overline{x}t)\,\Lambda_T(\overline{x}^{-1})\,\|x\|\,S_\Lambda([w,0],x\overline{x}r).$$

Moreover, if $u \in U(L)_p$ then

$$S_{\Lambda,p}(u,r) = 1 + \sum_{j=1}^{\infty} q^j \beta_1^*(r/p^j),$$

and if $u = [w, -w\overline{w}D/4] \in U_p$ and $a = max(0, -ord_\pi(w)) > 0$, then

$$S_{\Lambda,p}(u,r) = \Lambda_T^*(\pi) q^{a-1} p^{-3k/2} \beta_1(r/p^{a-1}) \; + \sum_{j=a}^{\infty} q^j \beta_1^*(r/p^j),$$

which is zero unless $a < ord_p(r)+1$. In particular, if $r \in \mathbf{Z}_p^*$, then

$$S_{\Lambda,p}([w,-w\overline{w}D/4],r) = \begin{cases} 1 - p^{-3k} & \text{if } w \in O_p, \\ \Lambda_T^*(\pi)p^{-3k/2}\left(1 - \Lambda_T^*(\pi)p^{-3k/2}\right) & \text{if } w \in \pi^{-1}O_p^*, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to the proof of the previous proposition (the case of p inert). Indeed, the first assertion about the value of the Siegel function at $x[w,t]$ is a reformulation of Lemma 5.1 which allows us to restrict our attention to the values of the Siegel function on elements of the form $u = [w, -w\overline{w}D/4]$. Thus we must evaluate

$$S_{\Lambda,p}(u,r) = \int_{\mathbf{Q}_p} \Lambda(\delta(\iota[w, t - w\overline{w}D/4]))\lambda_p(-rt)dt,$$

which we will do by using the explicit formula for $\delta(\iota u)$.

As before, we apply Lemma 5.4.4 which asserts that $\delta(\iota u) = d(1, \pi^{\alpha(u)})$ where

$$\alpha([w, t - w\overline{w}D/4]) = max(0, -ord_\pi(w), -ord_\pi(t - w\overline{w}\omega_p)),$$

where $\omega_p = (D/2+\tau)/2$ is our chosen generator for the maximal ideal of O_p.

Observe that for any nonnegative integer j we have

$$\pi^{2j}O_p = p^j\mathbf{Z}_p + p^j\mathbf{Z}_p\omega_p, \qquad \pi^{2j+1}O_p = p^{j+1}\mathbf{Z}_p + p^j\mathbf{Z}_p\omega_p.$$

(This follows easily from the fact that $\pi O_p = \overline{\pi}O_p$ and $\pi^2 O_p = pO_p$). Thus, if $t \in p^{-b}\mathbf{Z}_p^*$ and $w \in \pi^{-a}O_p^*$, then

$$\alpha([w, t - w\overline{w}D/4]) = max(0, a, -ord_\pi(p^{-b}\mathbf{Z}_p^* - p^{-a}\mathbf{Z}_p^*\omega_p)$$

$$= \begin{cases} 0 & \text{if } a, b \le 0 \ , \\ 2a-1 & \text{if } a > max(0,b) \ , \\ 2b & \text{if } b \ge max(0,a) \ . \end{cases}$$

If $w \in O_p$, then $a = 0$. So $\alpha([w, t - w\overline{w}D/4]) = max(0, 2b)$. This implies

$$S_{\Lambda,p}(u,r) = \int_{\mathbf{Z}_p} \Lambda_T(1)\lambda_p(-rt)dt + \sum_{j=1}^{\infty} \int_{p^{-j}\mathbf{Z}_p^*} \Lambda_T(\pi^{2j})\lambda_p(-rt)dt.$$

Recall that $\Lambda_T(\pi) = \Lambda_T^*(\pi)\|\pi\|^{-3k/2}$ and $\Lambda_T^*(\pi) = \pm 1$ (§1.7). Using these facts we can write

$$S_{\Lambda,p}(u,r) = 1 + \sum_{j=1}^{\infty} p^{3kj}\beta_{p^{-j}}^*(r),$$

which simplifies to the formula in the Proposition, since $\beta_{p^{-j}}^*(r) = p^j\beta_1^*(r/p^j)$ and $q = p^{-(3k-1)}$.

If $w \in \pi^{-a}O_p^*$ with $a < 0$, then $\alpha([w, t - w\overline{w}D/4]) = max(2b, 2a-1)$, so that

$$S_{\Lambda,p}(u,r) = \int_{p^{-(2a-1)}\mathbf{Z}_p} \Lambda_T(\pi^{2a-1})\lambda_p(-rt)dt + \sum_{j=a}^{\infty} \int_{p^{-j}\mathbf{Z}_p^*} \Lambda_T(\pi^{2j})\lambda_p(-rt)dt$$

$$= \Lambda_T^*(\pi)p^{(3k/2)(2a-1)}\beta_{p^{-(2a-1)}}(r) + \sum_{j=a}^{\infty} p^{3kj}\beta_{p^{-j}}^*(r),$$

which simplifies to the formula given in the proposition. **Q.E.D.**

5.5. Evaluation of the local Siegel function at split primes

Throughout this section, p will denote a rational prime that splits in K. In this case, K_p is a product of fields $K_p \cong \mathbf{Q}_p^2$ and the local group G_p is isomorphic to $GL_3(\mathbf{Q}_p) \times \mathbf{Q}_p^*$. We will prove the existence of a local Iwasawa decomposition for G_p, and will develop an explicit formula for the Iwasawa function δ (c.f. §1.5) restricted to ιU_p. This will be done by first proving a corresponding result for $GL_3(\mathbf{Q}_p)$ and then pulling the result back to G_p using an explicit isomorphism. We will then use the formula for the Iwasawa function to obtain a formula for the local Siegel function, $S_{\Lambda,p}(m_p, r)$, for all $m_p \in M_p$ and all $r \in \mathbf{Q}_p^*$.

5.5.1. The structure of K_p and G_p

We begin by reviewing some of the elementary properties of the ring K_p and the group G_p.

Since there are two distinct primes of K which divide p, $K_p = K \otimes \mathbf{Q}_p$ is isomorphic to $\mathbf{Q}_p^2$, so there exist idempotents e, $\overline{e}$ in K_p such that $e\overline{e} = 0$, $e + \overline{e} = 1$, and $K_p = \mathbf{Q}_p e + \mathbf{Q}_p \overline{e}$. Let $\pi = pe + \overline{e}$, then $p = \pi\overline{\pi}$ and K_p^*/O_p^* is generated by π and $\overline{\pi}$. Let $\tau = \sqrt{-D}$, then

τ generates K_p over $\mathbf{Q}_p$ and there is a $\tau_1 \in \mathbf{Q}_p^*$ such that $\tau = \tau_1 e - \tau_1 \overline{e}$, since $\overline{\tau} = -\tau$. Observe that

$$GL_3(K_p) = GL_3(\mathbf{Q}_p)e \oplus GL_3(\mathbf{Q}_p)\overline{e},$$

and let $j_1, j_2 : GL_3(K_p) \to GL_3(\mathbf{Q}_p)$ be the projections onto the first and second factors of this decomposition, so $g = j_1(g)e + j_2(g)\overline{e}$ and $j_2(g) = j_1(\overline{g})$.

Recall that G_p is the group of unitary similitudes in $GL_3(K_p)$ with respect to the hermitian matrix R defined (in §1.1) by

$$R = \begin{pmatrix} 0 & 0 & \tau \\ 0 & -D & 0 \\ -\tau & 0 & 0 \end{pmatrix};$$

let R_p be its image in $M_3(K_p)$, and let $R_1 = j_1(R)$, $R_2 = j_2(R)$, so $R_p = R_1 e + R_2 \overline{e}$. Since R is hermitian (${}^t\overline{R} = R$), we see that $R_2 = {}^tR_1$. and it follows that

$$G_p = \left\{g = g_1 e + g_2 \overline{e} : g_1, g_2 \in GL_3(\mathbf{Q}_p), \quad {}^tg_2 R_1 g_1 = \mu(g) R_1, \quad \mu(g) \in \mathbf{Q}_p\right\}.$$

Lemma 5.5.1. *$GL_3(\mathbf{Q}_p) \times \mathbf{Q}_p^* \cong G_p$ and $GL_3(\mathbf{Z}_p) \times \mathbf{Z}_p^* \cong G(L)_p$ via the isomorphism ϕ defined by*

$$\phi(g_1, \mu) = g_1 e + \left(\mu R_1 g_1^{-1} R_1^{-1}\right) \overline{e}.$$

The inverse of ϕ is $\phi^{-1}(g) = (j_1(g), \mu(g))$ where $j_1 : GL_3(K_p) \to GL_3(\mathbf{Q}_p)$ is defined by $g = j_1(g)e + j_1(\overline{g})\overline{e}$.

Proof: Except for the statements relative to the stabilizer of the lattice, this is clear from the preceding remarks.

Recall that $L_p = \mathbf{Z}_p^3$. Thus, $g = g_1 e + g_2 \overline{e}$ stabilizes L_p if and only if (i) $g_1 \in GL_3(\mathbf{Z}_p)$ and (ii) $\mu(g) R_1 g_1^{-1} R_1^{-1} \in GL_3(\mathbf{Z}_p)$. Since p is not ramified, D and τ are units of $\mathbf{Z}_p$, and so $R_1 \in GL_3(\mathbf{Z}_p)$. So condition (ii) holds iff $\mu(g) \in \mathbf{Z}_p^*$. **Q.E.D.**

The closure U_p of the unipotent subgroup $U_{\mathbf{Q}}$ in G_p is isomorphic, under the isomorphism of the previous lemma, to the group $\mathcal{U}_p$ of upper triangular unipotent matrices in $GL_3(\mathbf{Q}_p)$. Indeed, $U_p = \{[w, t] : w \in K_p, t \in \mathbf{Q}_p\}$, and for all $u = [w_1 e + w_2 \overline{e}, t] \in U_p$ we have $\phi^{-1}(u) = (g_1, 1)$ where

$$g_1 = \begin{pmatrix} 1 & \tau_1 w_2 & t + \tau_1 w_1 w_2/2 \\ 0 & 1 & w_1 \\ 0 & 0 & 1 \end{pmatrix},$$

where we recall that $\tau_1 \in \mathbf{Q}_p$ is the element such that $\tau = \tau_1 e - \tau_1 \overline{e}$.

The completion D_p of the maximal torus $D_{\mathbf{Q}}$ is isomorphic to $\mathcal{D}_p \times \mathbf{Q}_p^*$, where $\mathcal{D}_p$ is the group of diagonal matrices in $GL_3(\mathbf{Q}_p)$. Indeed, $D_p = \left\{d(z, x) : z, x \in K_p^*\right\}$, and $\mu(d(z, x)) = z\overline{z} = z_1 z_2$ for $z = z_1 e + z_2 \overline{e}$. Thus $\mu(D_p) \cong \mathbf{G_m}(\mathbf{Q}_p)$ and $ker(\mu|D) \cong \mathcal{D}_p$.

5.5.2. The Iwasawa decomposition of G_p with respect to $G(L)_p$

The Iwasawa decomposition for G_p with respect to $G(L)_p = G_p(L_p)$ for $L_p = O_p^3$ follows easily from the Iwasawa decomposition for $GL_3(\mathbf{Q}_p)$ with respect to $GL_3(\mathbf{Z}_p)$, as seen in the next lemma.

Lemma 5.5.2. *Let p be a prime which splits in K and let $L_p = O_p^3$. Then $G_p = P_p G_p(L_p)$.*

Proof: Recall that $P_p = D_p U_p$ where D and U are as above. Let $g = g_1 e + g_2 \bar{e} \in G_p$, then $g_1 \in GL_3(\mathbf{Q}_p)$. So there exists an upper triangular matrix $h_1 \in GL_3(\mathbf{Q}_p)$ such that $g_1 = h_1 \lambda_1$ for some $\lambda_1 \in GL_3(\mathbf{Z}_p)$. Let $h = h_1 e + h_2 \bar{e}$ where $h_2 = R_1 h_1^{-1} R_1^{-1}$, then $h \in P_p$. Similarly, let $\lambda = \lambda_1 e + \lambda_2 \bar{e}$ with $\lambda_2 = R_1 \lambda_1^{-1} R_1^{-1}$. Since p is not ramified in K, $R_1 \in Gl_3(\mathbf{Z}_p)$ so $\lambda \in G(L)_p$. Thus, $\tilde{g} = h^{-1} g \lambda$ is an element of G_p with $\tilde{g}_1 = 1$. Thus g_2 is a scalar matrix μI_3 and so $g = \mu^{-1} h \lambda$ is the desired Iwasawa decomposition of g. **Q.E.D.**

5.5.3. An explicit formula for the Iwasawa function of $GL_3(\mathbf{Q}_p)$

The goal of this section is to find an explicit formula for the local Iwasawa function δ_0 of $GL_3(\mathbf{Q}_p)$ restricted to the subset $\iota \mathcal{U}_p$. We will use the results of this section to find an explicit formula for the local Iwasawa function on G_p restricted to ιU_p.

Let $\mathcal{P}_p$ be the maximal parabolic subgroup of upper triangular matrices in $GL_3(\mathbf{Q}_p)$ and let $\mathcal{U}_p$ be its unipotent radical and $\mathcal{D}_p$ its maximal torus. In this section, we obtain an explicit formula for the Iwasawa function δ_0 restricted to the opposite subgroup $\mathcal{U}_p^-$. This function δ_0 maps $GL_3(\mathbf{Q}_p)/GL_3(\mathbf{Z}_p)$ to $\mathcal{D}_p/\mathcal{D}(\mathbf{Z}_p)$ and is defined by $g \in \delta_0(g)\mathcal{U}_p GL_3(\mathbf{Z}_p)$. The opposite subgroup $\mathcal{U}^-$ is the unipotent radical of the unique maximal parabolic subgroup of $GL_3(\mathbf{Q}_p)$, different from $\mathcal{P}_p$, whose maximal torus is $\mathcal{D}_p$. Thus, $\mathcal{U}^- = \iota_0 \mathcal{U}_p \iota_0^{-1}$ where

$$\iota_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $w_i \in \mathbf{Q}_p$, let $u = [w_1, w_2, w_3]$ denote the following element of $\mathcal{U}_p$:

$$u = [w_1, w_2, w_3] = \begin{pmatrix} 1 & w_2 & w_3 \\ 0 & 1 & w_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group law of $\mathcal{U}_p$ with respect to this parameterization is given by

$$[w_1, w_2, w_3][l_1, l_2, l_3] = [w_1 + l_1, w_2 + l_2, w_3 + l_3 + w_2 l_1].$$

We want to calculate $f(w_1, w_2, w_3) = \delta_0(\iota_0 [w_1, w_2, w_3] \iota_0^{-1})$. Since f is constant on the cosets $u\mathcal{U}(\mathbf{Z}_p)$ we may assume, for the purpose of calculation, that $w_i \in p^{-a_i}\mathbf{Z}_p^*$ with $a_i \geq 0$, for $i = 1, 2, 3$, by translating u within its $\mathcal{U}(\mathbf{Z}_p)$ coset.

For $x_i \in \mathbf{Z}_p^*$, let $x = diag(x_1, x_2, x_3) \in \mathcal{D}(\mathbf{Z}_p)$. Such elements act by conjugation of $\mathcal{U}_p/\mathcal{U}(\mathbf{Z}_p)$ by

$$x[w_1, w_2, w_3]x^{-1} = [(x_2/x_3)w_1, (x_1/x_2)w_2, (x_1/x_3)w_3].$$

Since $\delta_0(g)$ is invariant under the action $g \mapsto xgx^{-1}$ for $x \in \mathcal{D}(\mathbf{Z}_p)$, we may further assume for the purpose of calculation that $w_1 = p^{-a}$, $w_2 = p^{-b}$ for some $a, b \geq 0$.

Let y_1 and y_2 be the diagonal matrices with entries $(1, p, p^{-1})$ and $(p, p^{-1}, 1)$ respectively. These generate the subgroup of all $d \in \mathcal{D}_p/\mathcal{D}(\mathbf{Z}_p)$ such that $det(d) \in \mathbf{Z}_p^*$, and so for all $u \in \mathcal{U}_p$, $\delta_0(\iota_0 u \iota_0^{-1}) = y_1^a y_2^b$ for some $a, b \in \mathbf{Z}$, depending on u.

Lemma 5.5.3. *Let $u = [w_1, w_2, w_3] \in \mathcal{U}(\mathbf{Q}_p)$, such that*

$$w_1 \in p^{-a}\mathbf{Z}_p^*, \qquad w_2 \in p^{-b}\mathbf{Z}_p^*, \qquad w_3 \in p^{-c}\mathbf{Z}_p^*,$$

for a, b, c integers, $a, b \geq 0$. If $a + b \neq c$ then

$$\delta_0(\iota_0 u \iota_0^{-1}) = y_1^{max(b,c)} y_2^{max(a+b,c)}.$$

If $a + b = c$, let $w_4 = w_1 w_2 - w_3$ and let $d = -ord_p(w_4)$. Then $d \leq a + b$ and

$$\delta_0(\iota_0 u \iota_0^{-1}) = y_1^{a+b} y_2^{max(a,d)}.$$

Proof: For each of the cases of the lemma, we provide matrices $m \in \mathcal{P}_p$ and $\lambda \in GL_3(\mathbf{Z}_p)$ such that $m\lambda = \iota_0 u \iota_0^{-1}$ for u given in terms of w_1, w_2, w_3, with the w_i restricted as above. As noted above, after conjugation by an appropriate element of $\mathcal{D}(\mathbf{Z}_p)$, we may assume that $w_1 = p^{-a}$, $w_2 = p^{-b}$ and $w_3 = vp^{-c}$ where $a, b \geq 0$, $c \in \mathbf{Z}$, and $v \in \mathbf{Z}_p^*$. Then

$$\iota_0 u \iota_0^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ p^{-a} & 1 & 0 \\ vp^{-c} & p^{-b} & 1 \end{pmatrix}.$$

i) Assume first that $c < a + b$

Let $w_3 = vp^{-c}$ where $v \in \mathbf{Z}_p^*$ and let $v_1 = 1 - vp^{a+b-c} \in \mathbf{Z}_p^*$. Then there are two subcases depending on whether $c > b$ or not:

$$\iota_0 u \iota_0^{-1} = \begin{cases} \begin{pmatrix} p^{a+b}/v_1 & -p^{c-b}/v & 1 \\ 0 & -p^{c-(a+b)}v_1/v & p^{-a} \\ 0 & 0 & vp^{-c} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p^b/v_1 \\ 1 & p^{c-b}/v & p^c/v \end{pmatrix} & \text{if } c \geq b \;, \\ \begin{pmatrix} p^{a+b}/v_1 & 1 & 0 \\ 0 & v_1 p^{-a} & 1 \\ 0 & 0 & p^{-b} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -p^{a+b}/v_1 \\ vp^{b-c} & 1 & p^b \end{pmatrix} & \text{if } c \leq b \;. \end{cases}$$

ii) Now assume that $c > a + b$

Let $w_3 = vp^{-c}$ where $v \in \mathbf{Z}_p^*$ and let $v_2 = p^{c-(a+b)} - v \in \mathbf{Z}_p^*$. Then

$$\iota_0 u \iota_0^{-1} = \begin{pmatrix} p^c/v_2 & -p^{c-b}/v & 1 \\ 0 & -v_2/v & p^{-a} \\ 0 & 0 & vp^{-c} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -p^{c-a}/v_2 \\ 1 & p^{c-b}/v & p^c/v \end{pmatrix}.$$

iii) Finally, we consider the case $c = a + b$

In this case, we let $w_4 = w_1 w_2 - w_3 = p^{-(a+b)} - w_3 = v_3 p^{-d}$ for some $v_3 \in \mathbf{Z}_p^*$ and some $d \leq a + b$, $d \in \mathbf{Z}$. Then $v = 1 - v_3 p^{a+b-d}$ and there are two subcases (depending on whether $d < a$ or not) and we find that $\iota_0 u \iota_0^{-1} =$

$$\begin{cases} \begin{pmatrix} p^a & -p^{a+b}/v & 1 \\ 0 & -p^b/v & p^{-a} \\ 0 & 0 & vp^{-(a+b)} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & v_3 p^{a-d} & 1 \\ 1 & p^a/v & p^{a+b}/v \end{pmatrix} & \text{if } d \le a \ , \\ \begin{pmatrix} p^d/v_3 & -p^a/v & 1 \\ 0 & -p^{a+b-d}v_3/v & p^{-a} \\ 0 & 0 & vp^{-(a+b)} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p^{d-a}/v_3 \\ 1 & p^a/v & p^{a+b}/v \end{pmatrix} & \text{if } a \le d \le a+b \ . \end{cases}$$

Q.E.D.

5.5.4. Evaluation of the Iwasawa function on ιU_p

In this section, we use the explicit Iwasawa decompositions just calculated for a subset of $\iota\mathcal{U}_p \subset GL_3(\mathbf{Q}_p)$ to obtain explicit Iwasawa decompositions for a subset of $\iota U_p \subset G_p$.

Lemma 5.5.4. *Let $w \in \pi^{-a}\overline{\pi}^{-b}O_p^*$ with $a, b \ge 0$, and let $t \in p^{-c}\mathbf{Z}_p^*$. If $c \ne a+b$, then*

$$\Lambda\delta(\iota[w, t - w\overline{w}\tau_1/2]) = \chi_1(\pi)^{max(b,c)}\chi_1(\overline{\pi})^{max(c,a+b)}.$$

If, however, $c = a+b$, let $d = -ord_p(t - w\overline{w}\tau_1)$; then

$$\Lambda\delta(\iota[w, t - w\overline{w}\tau_1/2]) = \chi_1(\pi)^{a+b}\chi_1(\overline{\pi})^{max(a,d)},$$

where $\chi_1(x) = \chi_1^*(x)\|x\|^{3k/2}$, and χ_1^* is the unramified, unitary Hecke character of weight $3k$ given by $\chi_1^* = \Lambda_Z^2\Lambda_T^{*3} \in \mathcal{H}_{3k}^*(K)$ (c.f. §1.7).

Proof: This can be proved using the isomorphism ϕ of Lemma 5.5.1. Indeed, let

$$G'_p = \{g \in G_p : \mu(g) = 1\},$$

be the unitary subgroup of G_p. Then the map

$$\phi_1 : GL_3(\mathbf{Q}_p) \to G'_p, \qquad g \mapsto \phi(g, 1), \qquad (\S 5.5.1),$$

is an isomorphism whose restriction to $\mathcal{U}_p$ is given by

$$\phi_1([w_1, w_2\tau_1, w_3]) = [w_1 e + w_2\overline{e}, w_3 - w_1 w_2\tau_1/2],$$

and which maps y_1, y_2 to $y, \overline{y}$, respectively, where

$$y = \phi_1(y_1) = diag(\overline{\pi}, \pi/\overline{\pi}, \pi^{-1}) = d(\pi/\overline{\pi}, \pi^2/\overline{\pi}).$$

Let $f_w(t) = \Lambda\delta(\iota[w, t - w\overline{w}\tau_1/2])$ for $w = w_1 e + w_2\overline{e} \in K_p$ and $t \in \mathbf{Q}_p$; then

$$f_w(t) = \Lambda\delta\phi_1\phi_1^{-1}(\iota[w, t - w\overline{w}\tau_1/2]) = \Lambda\delta\phi_1(\iota_0[w_1, w_2, t])$$

$$= \Lambda\phi_1\delta_0(\iota_0[w_1, w_2, t]) = \Lambda\phi_1\delta_0(\iota_0[w_1, w_2, t]\iota_0^{-1})).$$

So a direct translation of Lemma 5.5.3 to this case provides a formula for $f_w(t)$, as follows: Let w and t be as in the statement of the lemma being proved, then if $c \neq a+b$, Lemma 5.5.3 implies that

$$f_w(t) = \Lambda(y)^{max(b,c)}\Lambda(\overline{y})^{max(c,a+b)}.$$

If, however, $c = a+b$, let $d = ord_p(t - w\overline{w}\tau_1)$; then

$$f_w(t) = \Lambda(y)^{a+b}\Lambda(\overline{y})^{max(a,d)},$$

where $y = d(\pi/\overline{\pi}, \pi^2/\overline{\pi})$. The values $\Lambda(y)$ are more conveniently expressed as special values of Hecke characters as follows: Recall (§1.7) that $\Lambda(d(z,x)) = \Lambda_Z(z)\Lambda_T^*(x)\|x\|^{3k/2}$ where $\Lambda_Z \in \mathcal{H}_0^*(K)$ and $\Lambda_T^* \in \mathcal{H}_k^*(K)$ are unramified Hecke characters. Thus,

$$\Lambda(y) = \Lambda_Z(\pi/\overline{\pi})\Lambda_T^*(\pi^2/\overline{\pi})p^{-3k/2} = \chi_1(\pi),$$

where $\chi_1(x) = \chi_1^*(x)\|x\|^{3k/2}$, and $\chi_1^* = \Lambda_Z^2\Lambda_T^{*3} \in \mathcal{H}_{3k}^*(K)$. **Q.E.D.**

5.5.5. Evaluation of the local Siegel function when p splits

We can now use the explicit evaluation of the local Iwasawa decomposition to evaluate the local Siegel function. The result in this case requires a rather involved calculation, but the idea behind it is simple. Recall that the local Siegel function is the continous function on $M_p \times \mathbf{Q}_p$ defined by

$$S_\Lambda(m, r) = \int_{N_p} \Lambda(\delta(\iota n m))\lambda_p(-r \cdot n)dn.$$

By applying Lemma 5.1, we can reduce to the case of evaluating the local Siegel function $S_\Lambda(m, r)$ on an element $m = [w, t_w]$ where we may choose t_w as we please for the purpose of calculation. Since we will apply Lemma 5.5.4 it will be convenient to choose $t_w = -w\overline{w}\tau_1/2$, and so we must evaluate

$$S_\Lambda([w, -w\overline{w}\tau_1/2], r) = \int_{N_p} \Lambda(\delta(\iota[w, t - w\overline{w}\tau_1/2]))\lambda_p(-rt)dt.$$

By Lemma 5.5.4, if w is in the subset $C_{a,b} = \pi^{-a}\overline{\pi}^{-b}O_p^*$ for $a, b > 0$, then the interesting part of the integrand,

$$f_w(t) = \Lambda(\delta(\iota[w, t - w\overline{w}\tau_1/2])),$$

is a locally constant function of t and by determining the subsets on which it is constant, we can evaluate the local Siegel function directly. The results of these calculations are stated in the following Proposition.

Proposition 5.5.5. *Let p be a prime which splits in K, Then for every $0 \neq r \in \mathbf{Z}_p$, $x \in T_p$, $[w,t] \in U_p$ we have*

$$S_{\Lambda,p}(x[w,t],r) = \lambda(rx\overline{x}t)\Lambda_T(\overline{x}^{-1})\|x\|S_\Lambda([w,0],x\overline{x}r).$$

Moreover, let $\pi = pe + \overline{e}$, $\tau = \tau_1 e - \tau_1\overline{e}$ (c.f. 5.5.1) $q = p^{-(3k-1)}$, and let β_1 and β_1^* be the exponential sums defined in §5.3. Then if

$$w \in C_{a,b} = \pi^{-a}\overline{\pi}^{-b}O_p^* + O_p, \qquad a,b \geq 0,$$

we have $S_{\Lambda,p}([w,0],r) =$

$$\lambda_p(rw\overline{w}\tau_1/2)\epsilon'_{a,b}(\chi_1,r) \;+\; \lambda_p(\pm rw\overline{w}\tau_1/2)\epsilon_{a,b}(r) \;+\; \overline{\lambda_p(rw\overline{w}\tau_1/2)\epsilon'_{b,a}(\chi_1,r)},$$

where $\chi_1(x) = \chi_1^*(x)\|x\|^{3k/2}$, and $\chi_1^* = \Lambda_Z^2\Lambda_T^{*3}$ is the unramified, unitary Hecke character of weight $3k$ constructed from the restrictions of Λ to the subtori Z and T (c.f. §1.7) and the constants $\epsilon_{a,b}$ are given by

$$\epsilon_{a,b}(r) = q^{a+b}\beta_1(r/p^{a+b}) + \sum_{c=a+b+1}^{\infty} q^c\beta_1^*(r/p^c),$$

which is zero if $ord_p(r) < a+b$ (so $\lambda_p(\pm rw\overline{w}\tau_1/2)\epsilon_{a,b}(r)$ is well-defined), and

$$\epsilon'_{a,b}(\chi_1,r) = \overline{\chi}_1(\pi)^a q^b \left(\beta_1(r/p^b) + \sum_{j=1}^{a}(p\chi_1(\pi))^j\,\beta_1^*(r/p^{j+b})\right) - q^{a+b}\beta_1(r/p^{a+b}),$$

which is zero if $a = 0$ or if $ord_p(r) < b$. From this formula, it follows that

$$S_\Lambda([\overline{w},-t]) = \overline{S_\Lambda([w,-t])},$$

for all $[w,t] \in U_p$, and that $S_{\Lambda,p}([w,-w\overline{w}\tau_1/2],r)$ is constant on the sets $C_{j,0}$ for all $j \geq 0$.

Proof: The first assertion, concerning the value of the local Siegel function on elements of the form $x[w,t]$, is a restriction of Lemma 5.1 to this case. It allows us to focus our attention on the values $S_\Lambda([w,t_w],r)$ where w ranges through K_p and we are free to choose t_w as we please for the purpose of calculation.

We prove the assertion concerning the value of $S_\Lambda([w,0],r)$ for $w \in C_{a,b} = \pi^a\overline{\pi}^bO_p^* + O_p$, in two steps. First we assume that $w \in \pi^{-a}\overline{\pi}^{-b}O_p^*$ so that we may apply Lemma 5.5.4. This takes care of all cases in which $a,b > 0$. Then we will reduce the general case $w \in C_{a,b}$ to this case by proving the last two assertions of the Proposition.

We will now evaluate the local Siegel function, under the assumption that $w \in \pi^{-a}\overline{\pi}^{-b}O_p^*$. We will evaluate it at the point $[w,-w\overline{w}\tau_1/2]$ and will use Lemma 5.1 to obtain the value at $[w,0]$. Recall that the local Siegel function is defined at this point by

$$S_{\Lambda,p}([w,-w\overline{w}\tau_1/2],r) = \int_{N_p} f_w(t)\lambda_p(-rt)dt,$$

where $f_w(t) = \Lambda\delta(\iota[w, t - w\overline{w}\tau_1/2])$ is the interesting part of the integrand. Lemma 5.5.4 implies that if $w \in \pi^{-a}\overline{\pi}^{-b}O_p^*$, then $f_w(t)$ is constant on the sets $p^{-b}\mathbf{Z}_p$ and $p^{-c}\mathbf{Z}_p^*$ for all $c > b$ except for $c = a + b$, and its values on these sets are given for $t \in p^{-c}\mathbf{Z}_p^*$ as follows when $c \neq a + b$:

$$f_w(t) = \begin{cases} \chi_1(\pi)^b \chi_1(\overline{\pi})^{a+b} & \text{if } c \leq b \ , \\ \chi_1(\pi)^c \chi_1(\overline{\pi})^{a+b} & \text{if } b < c \leq a + b \ , \\ \chi_1(\pi)^c \chi_1(\overline{\pi})^c & \text{if } c > a + b \ . \end{cases}$$

If $t \in p^{-(a+b)}\mathbf{Z}_p^*$, then the situation is more complicated. Indeed, Lemma 5.5.4 implies that $f_w(t)$ is constant on the sets

$$X_a(w) = w\overline{w}\tau_1 + p^{-a}\mathbf{Z}_p,$$

$$X_j^*(w) = w\overline{w}\tau_1 + p^{-j}\mathbf{Z}_p^* \qquad j = a+1, \ldots, a+b-1, \text{ and}$$

$$p^{-a+b}\mathbf{Z}_p^* - X_{a+b-1}(w).$$

Moreover, the values that $f_w(t)$ takes on these subsets are given by

$$f_w(t) = \begin{cases} \chi_1(\pi)^{a+b}\chi_1(\overline{\pi})^a & \text{if } t \in X_a(w) \ , \\ \chi_1(\pi)^{a+b}\chi_1(\overline{\pi})^d & \text{if } t \in X_j(w), \ a+1 \leq d \leq a+b-1 \ , \\ \chi_1(\pi)^{a+b}\chi_1(\overline{\pi})^{a+b} & \text{if } t \in p^{-a+b}\mathbf{Z}_p^* - X_{a+b-1}(w) \ . \end{cases}$$

Thus, if we decompose the integral into a sum of integrals over the regions on which $f_w(t)$ is constant, we will obtain the formulas given in the Proposition.

We now perform these calculations. First, the integral over the region $-ord_p(t) < a+b$ is easily seen to be

$$I_1 = \Lambda(y)^b \Lambda(\overline{y})^{a+b} \beta_{p^{-b}}(r) + \sum_{c=b+1}^{a+b-1} \Lambda(y)^c \Lambda(\overline{y})^{a+b} \beta_{p^{-c}}^*(r)$$

$$= \overline{\chi}_1(\pi)^a q^b \left(\beta_1(r/p^b) + \sum_{j=1}^{a-1} (p\chi_1(\pi))^j \beta_1^*(r/p^{j+b}) \right),$$

and the integral over the region $-ord_p(t) > a + b$ yields

$$I_2 = \sum_{c=a+b+1}^{\infty} \Lambda(y\overline{y})^c \beta_{p^{-c}}^*(r) = \sum_{c=a+b+1}^{\infty} q^c \beta_1^*(r/p^c).$$

This leaves the region $p^{-(a+b)}\mathbf{Z}_p^*$ to be considered. The integral over the region

$$X_{a+b-1}(w) = \{w : -ord_p(t - w\overline{w}\tau_1) < a + b\}$$

is easily seen to be

$$I_3 = \lambda_p(-rw\overline{w}\tau_1) \left(\Lambda(y)^{a+b}\Lambda(\overline{y})^a \beta_{p^{-a}}(r) + \sum_{d=a+1}^{a+b-1} \Lambda(y)^{a+b}\Lambda(\overline{y})^d \beta_{p^{-d}}^*(r) \right)$$

$$= \lambda_p(-rw\overline{w}\tau_1)\chi_1(\pi)^b q^a \left(\beta_1(r/p^a) + \sum_{j=a}^{b-1} (p\overline{\chi}_1(\pi))^j \beta_1^*(r/p^{j+a}) \right).$$

This leaves only the region $p^{-(a+b)}\mathbf{Z}_p^* - X_{a+b-1}(w)$ whose contribution to the integral is

$$I_4 = \Lambda(y\overline{y})^{a+b} \left(\beta^*_{p^{-(a+b)}}(r) - \lambda_p(-rw\overline{w}\tau_1)\beta_{p^{-(a+b)}}(r) \right)$$

$$= q^{a+b}\beta_1^*(r/p^{a+b}) - q^{a+b}\lambda_p(-rw\overline{w}\tau_1)p^{-1}\beta_1(rp/p^{a+b})$$

$$= q^{a+b}\beta_1^*(r/p^{a+b}) - q^{a+b}\lambda_p(-rw\overline{w}\tau_1)\beta_1(r/p^{a+b}) + q^{a+b}\lambda_p(-rw\overline{w}\tau_1)\beta_1^*(r/p^{a+b}).$$

Let I_4', I_4'', and I_4''' be the three terms of the last equation. Since $\beta_1(r/p^{a+b})$ vanishes unless $ord_p(r) \geq a+b$, we see that

$$I_4'' = -q^{a+b}\beta_1(r/p^{a+b}).$$

The Proposition follows (under the assumption $w \in \pi^{-a}\overline{\pi}^{-b}O_p^*$) upon observing that

$$S_{\Lambda,p}([w,0],r) = \lambda_p(rw\overline{w}\tau_1/2)\,(I_1 + I_2 + I_3 + I_4)\,,$$

where

$$I_1 + I_4' + I_4'' = \epsilon_{a,b,\chi_1}(r)\ ,$$

$$I_2 - I_4'' = \epsilon_{a,b}(r) = \lambda_p(-rw\overline{w}\tau_1)\,(I_2 - I_4'')\ ,$$

and

$$(I_3 + I_4''' + I_4'') = \lambda_p(-rw\overline{w}\tau_1)\overline{\epsilon_{b,a,\chi_1}(r)}\ .$$

This completes the proof in the case where $w \in \pi^{-a}\overline{\pi}^{-b}O_p^*$.

We must now treat the cases where $C_{a,b} \neq \pi^{-a}\overline{\pi}^{-b}O_p^*$ which are $C_{0,0}$, $C_{a,0}$, $C_{0,b}$ for $a, b > 0$. We will prove that the formulas of the Proposition hold for w in these sets by showing that $S_{\Lambda,p}([w, -w\overline{w}\tau_1/2], r)$ is constant on $C_{a,0}$ and observing that

$$\overline{S_\Lambda([w,t],r)} = S_\Lambda([\overline{w},-t],r),$$

which implies that $S_{\Lambda,p}([w, +w\overline{w}\tau_1/2], r)$ is constant on $C_{0,b}$.

Consider first the case of $w \in C_{0,0} = O_p$. Since $\epsilon_{0,0}(\chi_1, r) = 0$, we must show that

$$S_{\Lambda,p}([w, -w\overline{w}\tau_1/2], r) = \epsilon_{a,b}(r).$$

Since $U(L)_p = \{[w, t - w\overline{w}\tau_1/2] : w \in O_p, t \in \mathbf{Z}_p\}$ we see that

$$S_{\Lambda,p}([w, -w\overline{w}\tau_1/2], r) = S_{\Lambda,p}([0,0], r),$$

is constant on O_p, and since the Proposition holds for $w \in O_p^*$ it holds for $w \in O_p$.

Next, suppose $w \in C_{0,b}$ with $b > 0$; then since $\epsilon_{0,b}(\chi_1, r) = 0$, it will suffice to show that

$$S_{\Lambda,p}([w, w\overline{w}\tau_1/2], r) = \epsilon_{a,b}(r) + \overline{\epsilon}'_{b,0}(\chi_1, r),$$

and since this has been proved for $w \in \overline{\pi}^{-b}O_p^*$, we need only show that $S_{\Lambda,p}([w, w\overline{w}\tau_1/2], r)$ is constant on $C_{0,b}$. But this is easy. Let $w = w_1 e + w_2 \overline{e}$ with $w_1 \in O_p$, then

$$[w, w\overline{w}\tau_1/2] = [w_2\overline{e}, 0][w_1 e, 0],$$

and since $[w_1 e, 0] \in U(L)_p$ $S_{\Lambda,p}([w, w\overline{w}\tau_1/2], r) = S_{\Lambda,p}([w_2\overline{e}, 0], r)$ is independent of w_1, hence is constant on $C_{0,b}$.

The case of $C_{a,0}$ with $a > 0$ is similar, except that we show

$$S_{\Lambda,p}([w, -w\overline{w}\tau_1/2], r) = \epsilon'_{a,0}(\chi_1, r) + \epsilon_{a,b}(r),$$

using the fact that $[w, -w\overline{w}\tau_1/2] = [w_1 e, 0][w_2\overline{e}, 0]$. **Q.E.D.**

Corollary 5.5.6. *If $r \in \mathbf{Z}_p^*$ and $a, b > 0$, then $S_{\Lambda,p}([w, 0], r) =$*

$$\begin{cases} \lambda_p(rw\overline{w}\tau_1/2)\ \left(1 - p^{-3k}\right) & \text{if } w \in O_p, \\ \lambda_p(rw\overline{w}\tau_1/2)\ \overline{\chi}_1(\pi)^a\ \left(1 - \chi_1(\pi)\right) & \text{if } w \in C_{a,0}, \\ \lambda_p(-rw\overline{w}\tau_1/2)\ \overline{\chi}_1(\overline{\pi})^b\ \left(1 - \chi_1(\overline{\pi})\right) & \text{if } w \in C_{0,b}, \\ 0 & \text{if } w \in C_{a,b}, \end{cases}$$

where $\chi_1(x) = \left(\Lambda_Z^2 \Lambda_T^{*3}\right)(x)\|x\|^{3k/2}$, and $\tau_1 \in \mathbf{Q}_p$ is defined by $\tau_1 e - \tau_1 \overline{e} = \tau$, where $\tau = \sqrt{-D}$.

Proof: Let $a, b \geq 1$ and $r \in \mathbf{Z}_p^*$, and let $\epsilon_{a,b}(r)$ and $\epsilon'_{a,b}(r)$ be as in the Proposition. Then, a simple calculation shows that

$\epsilon_{a,b}(r) = 0$ unless $r \in p^{a+b}\mathbf{Z}_p$,

$\epsilon'_{a,b}(\chi_1, r) = 0$ unless $r \in p^b \mathbf{Z}_p$, and

$\overline{\epsilon'_{b,a}(\chi_1, r)} = 0$ unless $r \in p^a \mathbf{Z}_p$.

Therefore, by the Proposition, $S_{\Lambda,p}([w, \pm w\overline{w}\tau_1/2], r) = 0$ if $a, b \geq 1$. The other three cases are clear from the Proposition and the observation that $w\overline{w} = 0$ if $w \in \mathbf{Q}_p e$. **Q.E.D.**

CHAPTER 6

EULER PRODUCTS II: PRIMITIVE COMPONENTS

In this Chapter, we find a formula for the primitive components of the adelic theta coefficients of the Eisenstein series E_Λ associated to a Hecke character Λ of weight k on the maximal torus of $G_{\mathbf{A}}$. The method we employ is to compute the Siegel inner products (§3.3) $(E_{\Lambda,\nu}, \theta)$ of the νth adelic theta coefficient of E_Λ with the theta functions θ belonging to the Shintani eigenspaces V_ξ defined in §4.4.

6.1. Statement of the Theorem

To state the Theorem, we first need to recall Shintani's results on the structure of the graded ring of adelic theta functions and the associated system of coordinates for modular forms. These results were presented in Chapter 4 and are summarized in the §6.1.1. We then state the main theorem of this Chapter and describe its proof, which is completed in the remaining 3 sections.

6.1.1. The primitive components of an adelic theta coefficient

Recall that the adelic theta coefficients of modular forms belong to the space $V_{k,\nu}(L)$ of adelic theta functions. This space can be decomposed into the orthogonal direct sum of the subspace $V_{k,\nu}(L)^0$ of primitive theta functions and the subspace of imprimitive theta functions. As we have seen(§4.4) there is an orthogonal direct sum decomposition of the graded ring of theta functions into primitive and imprimitive subspaces

$$\bigoplus_\nu V_{k,\nu}(L) = \bigoplus_\nu \bigoplus_a (l(a) V_{k,\nu}(L)^0),$$

where a ranges over all integral ideals of K and the operators $l(a)$ are injections which increase the level of a theta function by the factor $\mathcal{N}(a)$:

$$l(a) : V_{k,\nu}(L) \to V_{k,\nu\mathcal{N}(a)}(L).$$

Moreover, there is a representation, l_1, of the group of ideals prime to νD on the subspace of primitive theta functions of level ν which decomposes the space of primitive theta functions into eigenspaces $V_{k,\nu}(L)^0_\kappa$ where the eigencharacters κ are "almost" Hecke characters. More precisely, if we let κ^* be the map on ideals prime to νD defined by

$$\kappa^*(a) = \kappa(a/\overline{a}) \prod_{p\ inert} (-1)^{ord_p(a)},$$

then κ^* is a Hecke character of conductor C_ν dividing $4\nu D$. There is also a commuting family of projection operators l_π indexed by the ramified primes of K which operate on these eigenspaces and their eigenvalues are $\{0,1\}$. Thus, the eigenvalues of a simultaneous eigenfunction of the family of operators l_π can be specified by the set Σ of primes for which the eigenvalue of l_π is 1. In this way we obtain a decomposition of each of the eigenspaces of Shintani's representation into eigenspaces of the ramified prime projectors:

$$V_{k,\nu}(L)^0_\kappa = \bigoplus_\Sigma V_{k,\nu}(L)^0_{\kappa,\Sigma}.$$

To simplify the notation, we let Ξ be the set of all triples $\xi = (\nu, \kappa^*, \Sigma)$ such that the eigenspace $V_{k,\nu}(L)^0_{\kappa,\Sigma}$ is non-zero and we let V_ξ denote this eigenspace. Also, let Ξ_ν be the set of triples whose first component is ν. Using this notation, we can write the graded ring of theta functions as a direct sum of images of the eigenspaces V_ξ under the operators $l(a)$ for integral ideals a:

$$\bigoplus_\nu V_{k,\nu}(L) = \bigoplus_{\xi\in\Xi}\bigoplus_a (l(a)V_\xi).$$

If we choose an orthonormal basis $\mathcal{B}_\xi$ for each $\xi \in \Xi$, and let $\mathcal{B}$ be the union of the bases $\mathcal{B}_\xi$ then the adelic theta expansion of any modular form F can be written uniquely in the form, which we call the Shintani form of the adelic theta expansion of F:

$$F = \sum_{\xi\in\Xi}\sum_{\theta\in\mathcal{B}_\xi} Z_\theta \cdot \theta, \qquad Z_\theta\cdot\theta = \sum_a z_\theta(a) l(a)\theta,$$

where a ranges over all integral ideals and z_θ is a complex valued funtion on the monoid of integral ideals in K. We have seen that the outer two summations are orthogonal, and so the constants $z_\theta(1)$ for $\theta \in \mathcal{B}_\xi$, $\xi = (\nu, \kappa^*, \Sigma)$ can be found by evaluating a Siegel inner product $z_\theta(1) = (\theta, F_\nu)/(\theta,\theta)$. Thus, if F^0_ν denotes the primitive component of the adelic theta coefficient F_ν, we see that

$$F_\nu = \sum_{\xi\in\Xi_\nu}\sum_{\theta\in\mathcal{B}_\xi}\frac{(\theta, F_\nu)}{(\theta,\theta)}\theta.$$

6.1.2. The Main Theorem

Our Main Theorem asserts that $E_{\Lambda,\nu}$ restricted to V_ξ is dual to a certain linear functional:

$$(E_{\Lambda,\nu}, \theta) = \overline{L_\xi(k,\Lambda)}\, l_\Lambda(\theta), \qquad \forall\theta \in V_\xi,$$

where $L_\xi(k,\Lambda)$ is an explicit monomial of Hecke and Dirichlet L-series depending on Λ and $\xi \in \Xi_\nu$ and l_Λ is a simple linear functional on V_ξ.

More precisely, l_Λ is the linear functional on V_ξ defined by

$$l_\Lambda(\theta) = \sum_{\beta\in CL_K}\Lambda(\beta^{-1})\tilde\theta(\beta),$$

where $CL_K = (T_{\mathbf{Q}}T_\infty \cap T_f)\backslash T_f/T(L)_f$ is the ideal class group of K and the function $\tilde\theta(m) = j_{k,\nu}(m)^{-1}\theta(m)$ is the normalized value of the theta function (c.f. §3.1).

Theorem 6.1. *Let Λ be a Hecke character of weight k on the maximal torus of $G_{\mathbf{A}}$, let E_Λ be the associated Eisenstein series on $G_{\mathbf{A}}$, and let ν be a positive integer. Then the primitive component of the νth adelic theta coefficient of the Eisenstein series, is given by*

$$E^0_{\Lambda,\nu} = \sum_{\xi \in \Xi_\nu} L_\xi(k,\Lambda)\theta_\xi,$$

where $\theta_\xi \in V_\xi$ is the element dual to the linear operator l_Λ, defined above, that is

$$(\theta_\xi, \theta) = l_\Lambda(\theta) \qquad \forall \theta \in V_\xi,$$

and the complex number $L_\xi(k,\Lambda)$ is given, for $\xi = (\nu, \kappa^*, \Sigma)$ by the following monomial of special values of Hecke and Dirichlet L-series (c.f. §1.6):

$$L_\xi(k,\Lambda) = \frac{\alpha(k,\Lambda,\Sigma)}{4i\nu^2\sqrt{-D}^{3k+1}} \frac{L_K((3k-1)/2, \overline{\Lambda_Z^2\Lambda_T^*}\kappa^*)}{L_{\mathbf{Q}}(3k-1,\chi_K)\, L_K(3k/2, \overline{\Lambda_Z^2\Lambda_T^{*3}})},$$

where Λ_Z and Λ_T^* are the unramified, unitary Hecke characters of K associated to Λ as in §1.7, χ_K is the Dirichlet character of conductor D corresponding to the nontrivial element of the Galois group of $K/\mathbf{Q}$, and

$$\alpha(k,\Lambda,\Sigma) = \prod_{\pi \in \Sigma} \left(1 + \Lambda_T^*(\pi)\mathcal{N}(\pi)^{-(3k/2-1)}\right).$$

Moreover, $\overline{\Lambda_Z^2\Lambda_T^{*3}}$ is an unramified Hecke character of weight $-3k$, and $\overline{\Lambda_Z^2\Lambda_T^*}\kappa^*$ is a Hecke character of weight $-(3k-1)$ and conductor dividing $4\nu D$.

6.1.3. A sketch of the proof

We will develop this formula by using our knowledge of the factorization of the Siegel function S_Λ to evaluate the inner product

$$(E_{\Lambda,\nu}, \theta) = \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \overline{E_{\Lambda,\nu}}(m)\theta(m)dm,$$

"locally" for all $\theta \in V_\xi$ and all $\xi \in \Xi_\nu$ and we will show that it has the value

$$(E_{\Lambda,\nu}, \theta) = \overline{L_\xi(k,\Lambda)}\, l_\Lambda(\theta),$$

if $\theta \in V_\xi$. The local calculations are not as straightforward as in the case of tube domains because the integral does not split into a product of local integrals, and so we must use the definition of the adelic integral as a limit of integrals over compact subsets.

The calculation splits naturally into 3 parts, in which we calculate the contribution of the archimedean place, the nonarchimedean places prime to νD, and the nonarchimedean places dividing νD respectively.

In §6.2, we use the Siegel-Baily-Tsao-Karel integral for the Eisenstein series (§2.7) to obtain a corresponding integral for the inner product (§6.2.1)

$$(E_{\Lambda,\nu},\theta)=\int_{N_{\mathbf{Q}}T_{\mathbf{Q}}\backslash M_{\mathbf{A}}}\overline{S_\Lambda(m,\nu\|\delta(m_f)\|)}\theta(m)dm,$$

then we isolate the archimedean factors of the integrand (§6.2.2) and use the residue theorem to express the inner product as an integral which does not involve the archimedean place (§6.2.3)

$$(E_{\Lambda,\nu},\theta)=\epsilon_\infty\int_{J_K\times W_f}\mathcal{S}_{\Lambda,\theta}(a,w)dw\,da,$$

where J_K is the idele class group $(T_{\mathbf{Q}}T_\infty\cap T_F)\backslash T_f$, ϵ_∞ is an explicit constant, W is the quotient of U by its center N, and

$$\mathcal{S}_{\Lambda,\theta}(a,w)=\Lambda(a_f)^{-1}\overline{S_{\Lambda,f}(w,\nu a_f\overline{a}_f/\mathcal{N}(a_f))}\tilde{\theta}(aw).$$

In §6.3, we use local methods to reduce the inner product to an integral over the cartesian product of the spaces W_p for the finitely many primes p that divide νD. More precisely, we show that for every p that does not divide νD, and for every $w_S\in W_f$ such that $w_p=0$, one has

$$\int_{J_K\times W_p}\mathcal{S}_{\Lambda,\theta}(\beta,w_S+w_p)dw_p d\beta=\epsilon_p(k;\Lambda,\xi)\int_{J_K}\mathcal{S}_{\Lambda,\theta}(\beta,w_S)d\beta,$$

where the constants $\epsilon_p(k;\Lambda,\xi)$ are the local L-factors of $\overline{L_\xi(k,\Lambda)}$. That is

$$\overline{L_\xi(k,\Lambda)}=\epsilon_\infty\prod_p\epsilon_p(k;\Lambda,\xi).$$

We are then able to reduce the problem to one of evaluating an integral over a finite product space:

$$(E_{\Lambda,\nu},\theta)=\epsilon_\infty\prod_{p\nmid\nu D}\frac{\epsilon_p(k;\Lambda,\xi)}{1-p^{-3k}}\int_{J_K\times W_{\nu D}}\mathcal{S}_{\Lambda,\theta}(\beta,w)dw\,d\beta,$$

where $W_{\nu D}=\prod_{p|\nu D}W_p$. In §6.4, we use the fact that θ is a primitive eigenfunction to transform the inner product integral into the product of $l_\Lambda(\theta)$ by the explicit local L-factors associated to the primes dividing νD. More precisely, we show that

$$\int_{J_K\times W_{\nu D}}\mathcal{S}_{\Lambda,\theta}(\beta,w)dw\,d\beta=\left(\prod_{p\nmid\nu D}(1-p^{-3k})\right)\left(\prod_{p|\nu D}\epsilon_p(k;\Lambda,\xi)\right)l_\Lambda(\theta),$$

and this proves the Theorem.

6.1.4. The local factors of $\overline{L_\xi(k,\Lambda)}$

Before beginning the proof, we will describe explicitly the local L-factors of $\overline{L_\xi(k,\Lambda)}$ for $\xi=(\nu,\kappa^*,\Sigma)$ in the various cases depending on whether the prime p remains inert, ramifies, or splits in K and whether it divides νD or not and we will introduce some auxilliary characters that will be used in the proof.

Let the $\epsilon_p(k,\Lambda,\xi)$ denote the local p-factor of the L-series $L_\xi(k,\Lambda)$ which appears in the statement of Theorem 6.1. This L-series has the form

$$L_\xi(k,\Lambda) = \frac{1}{4i\nu^2\sqrt{-D}^{3k+1}} \prod_p \epsilon_p(k,\Lambda,\xi),$$

for certain explicitly defined p-factors, ϵ_p. Also let $\chi_1(x) = \Lambda_Z^2\Lambda_T^{*3}(x)\,\|x\|^{3k/2}$, $\chi_2(x) = \Lambda_T^{*2}\kappa^*(x)\,\|x\|^{-1/2}$, and $\chi_3(x) = \chi_1(x)\overline{\chi_2(x)} = \Lambda_Z^2\Lambda_T^*\overline{\kappa^*}(x)\|x\|^{(3k-1)/2}$ and let $\chi^*(x) = \chi(x)/|\chi(x)|$ for any character χ. Then, with this notation,

$$\overline{L_\xi(k,\Lambda)} = \frac{\overline{\alpha(k,\Lambda,\Sigma)}}{4i\nu^2\sqrt{-D}^{3k+1}} \frac{L_K((3k-1)/2,\chi_3^*)}{L_{\mathbf{Q}}(3k-1,\chi_K)\;L_K(3k/2,\chi_1^*)}$$

$$= \frac{1}{4i\nu^2\sqrt{-D}^{3k+1}} \prod_{\pi\in\Sigma}\left(1+\overline{\Lambda_T^*(\pi)}\mathcal{N}(\pi)^{-(3k/2-1)}\right)\prod_p \frac{\epsilon_{\mathbf{Q},p}(3k-1,\chi_K)\;\epsilon_{K,p}(3k/2,\chi_1^*)}{\epsilon_{K,p}((3k-1)/2,\chi_3^*)}.$$

Recall that a Hecke character with conductor C induces a character on the group of ideals of K prime to C, and that, for the purpose of defining the local L-factors, this map is extended to all ideals by requiring it to vanish on all ideals not relatively prime to C.

The case of p split

Let π be a prime of K dividing p. By the definition of ϵ_p in Theorem 6.1, the local L-factor is

$$\epsilon_p(k,\Lambda,\xi) = \frac{1-p^{-(3k-1)}}{1}\;\frac{1-\overline{\chi_1^*}(\pi)p^{-3k/2}}{1-\chi_3(\pi)p^{-(3k-1)/2}}\;\frac{1-\overline{\chi_1^*}(\overline{\pi})p^{-3k/2}}{1-\chi_3^*(\overline{\pi})p^{-(3k-1)/2}}.$$

If p divides ν, then, as a function on the group of ideals, $\kappa^*(\pi)=0$ and so $\chi_3^*(\pi)=0$ and the local factor has a simpler form:

$$\epsilon_p(k,\Lambda,\xi) = \left(1-p^{-(3k-1)}\right)\;\left(1-\overline{\chi_1^*}(\pi)p^{-3k/2}\right)\;\left(1-\overline{\chi_1^*}(\overline{\pi})p^{-3k/2}\right).$$

The case of p inert

If p remains inert in K, and p does not divide νD, and if π is the prime of K dividing p, then $\kappa^*(\pi)=\chi_K(\pi)=-1$ and $\Lambda_Z(\pi)=\Lambda_T^*(\pi)=1$. Thus,

$$\chi_1^*(\pi) = \Lambda_Z^2(\pi)\Lambda_T^{*3}(\pi) = 1, \qquad \chi_3^*(\pi) = \Lambda_Z^2(\pi)\Lambda_T^*(\pi)\overline{\kappa^*(\pi)} = -1,$$

and so the local L-factor is

$$\epsilon_p(k,\Lambda,\xi) = \frac{1+p^{-(3k-1)}}{1}\;\frac{1-p^{-3k}}{1+p^{-(3k-1)}} = 1-p^{-3k}.$$

If however, p divides νD, then $\kappa^*(\pi)=0$, so $\chi_3(\pi)=0$ and

$$\epsilon_p(k;\Lambda,\kappa)=\left(1+p^{-(3k-1)}\right)\left(1-p^{-3k}\right).$$

The case of p ramified

Finally, if p ramifies in K and π is the prime of K dividing p, then $\kappa^*(\pi)=0$, $\Lambda_Z(\pi)=\pm1$, $\Lambda_T^*(\pi)=\pm1$, and $\chi_K(\pi)=0$. So

$$\chi_1^*(\pi)=\Lambda_Z^2(\pi)\Lambda_T^{*3}(\pi)=\Lambda_T^*(\pi),\qquad \chi_3^*(\pi)=\Lambda_Z^2(\pi)\Lambda_T^*(\pi)\overline{\kappa^*(\pi)}=0.$$

Thus, if we distinguish between the cases $\pi\in\Sigma$ and $\pi\notin\Sigma$, and include the corresponding contribution of the factor $\alpha(k,\Lambda,\Sigma)$ we find that

$$\epsilon_p(k,\Lambda,\xi)=\begin{cases}\left(1-\Lambda_T^*(\pi)p^{-3k/2}\right)\left(1+\Lambda_T^*(\pi)p^{-(3k/2-1)}\right) & \text{if } \pi\in\Sigma\ ,\\ \left(1-\Lambda_T^*(\pi)p^{-3k/2}\right) & \text{if } \pi\notin\Sigma\ .\end{cases}$$

6.2. The contribution of the archimedean place

In this section, we determine the archimdian contribution to the inner product of the adelic theta coefficient $E_{\Lambda,\nu}$ with any theta function of level ν. We begin with the integral formula for the adelic theta coefficients which was obtained by applying the Siegel-Baily-Tsao-Karel approach (Lemma 2.7):

$$E_{\Lambda,\nu}(m)=\sum_{u\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}}S_\Lambda(um,\nu\|\delta(m_f)\|),$$

and we prove the following Proposition.

Proposition 6.2. *Let J_K be the idele class group $(T_{\mathbf{Q}}T_\infty\cap T_F)\backslash T_f$, let W be the quotient of U by its center N, and let $\theta\in V_{k,\nu}(L)$ be any theta function of level ν. Then*

$$(E_{\Lambda,\nu},\theta)=\epsilon_\infty\int_{J_K\times W_f}\mathcal{S}_{\Lambda,\theta}(a,w)\,dw\,da,$$

where the constant ϵ_∞ is

$$\epsilon_\infty=\frac{i}{4\nu^2\tau^{3k+1}},$$

and $\mathcal{S}_{\Lambda,\theta}$ is the continuous function defined on $J_K\times W_f$ by

$$\mathcal{S}_{\Lambda,\theta}(a,w)=\Lambda(a_f)^{-1}\overline{S_{\Lambda,f}(w,\nu\zeta(a_f))}\tilde{\theta}(aw),$$

where $S_{\Lambda,f}$ is the nonarchimedean component of the Siegel function (§5.1),

$$\tilde{\theta}(m)=j_{k,\nu}(m)^{-1}\theta(m)\qquad(\S3.1),$$

and $\zeta:K_f^*\to\mathbf{Z}_f^*$ is the map defined by $\zeta(a_f)=a_f\overline{a}_f/\mathcal{N}(a_f)$.

This will be proved in a series of 4 lemmas. The main idea is to transform the integrand in such a way that the holomorphicity property satisfied by the theta function (§3.1) will allow the archimedean contribution to the integral to be evaluated using the residue theorem.

6.2.1. Application of the Siegel-Baily-Tsao-Karel integral formula

The first step is to use the Siegel-Baily-Tsao-Karel formula for the Eisenstein series to obtain a simple integral formula for the inner product in question.

Lemma 6.2.1. *Let $\Lambda \in \mathcal{H}_k(D)$, let ν be a positive integer, and let $\theta \in V_{k,\nu}(L)$ be an adelic theta function, then*

$$(E_{\Lambda,\nu}, \theta) = \int_{T_{\mathbf{Q}}\backslash T_{\mathbf{A}}} \int_{N_{\mathbf{A}}\backslash U_{\mathbf{A}}} \overline{S_\Lambda(\dot{u}a, \nu\|a_f\|)}\theta(\dot{u}a)d\dot{u}da.$$

Proof: Unravelling the definitions of the inner product, and using the integral formula for the Eisenstein series given in Lemma 2.7, we find that

$$(E_{\Lambda,\nu}, \theta) = \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \left(\overline{E_{\Lambda,\nu}}\theta\right)(m)dm$$

$$= \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \left(\sum_{n\in N_{\mathbf{Q}}\cdot T_{\mathbf{Q}}\backslash M_{\mathbf{Q}}} \overline{S_\Lambda}(nm, \nu\|\delta(m_f)\|) \right) \theta(m)dm.$$

Since θ is left-$M_{\mathbf{Q}}$ invariant, we can bring θ into the inner sum

$$= \int_{M_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \sum_{n\in N_{\mathbf{Q}}\cdot T_{\mathbf{Q}}\backslash M_{\mathbf{Q}}} \overline{S_\Lambda(nm, \nu\|\delta(nm_f)\|)}\theta(nm)dm$$

$$= \int_{N_{\mathbf{Q}}\cdot T_{\mathbf{Q}}\backslash M_{\mathbf{A}}} \overline{S_\Lambda(m, \nu\|\delta(m)\|)}\theta(m)dm.$$

Now we can use the facts that the modulus of the action of $T_{\mathbf{Q}}$ on $U_{\mathbf{A}}$ is 1, and that M_A is the semidirect product of $T_{\mathbf{A}}$ and $U_{\mathbf{A}}$, to decompose the last integral into a multiple integral over the two factors of the semidirect product:

$$\int_{T_{\mathbf{Q}}\backslash T_{\mathbf{A}}} \int_{N_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \overline{S_\Lambda(ua, \nu\|a_f\|)}\theta(ua)duda,$$

and the lemma is completed by observing that the integrand is constant on right $N_{\mathbf{A}}$ cosets of $U_{\mathbf{A}}$ (c.f. §3.1,§5.1 for the $N_{\mathbf{A}}$ translation formulas of θ and S_Λ). This implies that the inner integral can be replaced by one over $N_{\mathbf{A}}\backslash U_{\mathbf{A}}$, since the compact quotient space $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}$ has measure 1, **Q.E.D.**

6.2.2. Isolation of the archimedean factors

We will now use the fact (§5.2) that the archimedean part of the Siegel function $S_\Lambda(m, \nu\|\delta(m_f)\|)$ is precisely the factor of automorphy $j_{k,\nu}(m)$ which appears in the holomorphicity part of the definition of an adelic theta function (§3.1) to show that the integrand $\left(\overline{S_\Lambda([w,t]a, \nu\|a_f\|)}\right)\ \theta([w,t]a)$, in the inner product formula for $(E_{\Lambda,\nu}, \theta)$ is, as a function of w_∞, the product of a holomorphic function by an exponential of the form $exp(-\pi c|w_\infty|^2)$.

Lemma 6.2.2. *Let $u = [w, 0] \in U_{\mathbf{A}}$ and $a \in T_{\mathbf{A}}$; then*

$$\left(\overline{S_\Lambda([w,0]a, \nu\|a_f\|)}\right)\ \theta([w,0]a) = \overline{c_1}\nu^{3k-1}\, h_1(a)\, h_2(u_f a_f)\, h_3(w_\infty, u_f a_f),$$

where the constant c_1 and the functions h_1, h_2, h_3 are given by

$$c_1 = \frac{(-2\pi i)^{3k}}{(3k-1)!},$$

$$h_1(a) = \|a\|^{3k} e^{-2\pi\nu\sqrt{D}\|a\|},$$

$$h_2(u_f a_f) = \|a_f\|^{-3k}\ \overline{\Lambda(a_f)\ S_{\Lambda,f}(a_f^{-1}u_f a_f, \nu a_f \overline{a}_f/\mathcal{N}(a_f))},\ \text{and}$$

$$h_3(w_\infty, u_f a_f) = \theta(w_\infty, u_f a_f)\ e^{-2\pi\|a_f\|\nu\sqrt{D}|w_\infty|^2}.$$

Furthermore, $\theta(w_\infty, u_f a_f) = \tilde{\theta}([w_\infty, 0]u_f a_f)$ (c.f. §3.1) is a holomorphic function of w_∞ for $u_f a_f$ fixed.

Proof: This lemma follows from straightforward calculations using Lemmas 5.1,5.2 and the definition of $\tilde{\theta}$. Let us first apply Lemmas 5.1,5.2 to obtain a partial evaluation of the Siegel function. Recall (§5.1) that the Siegel function factors into an archimedean and a nonarchimedean component:

$$S_\Lambda(ua, \nu\|a_f\|) = S_{\Lambda,\infty}((ua)_\infty, (\nu\|a_f\|)_\infty)\ S_{\Lambda,f}((ua)_f, (\nu\|a_f\|)_f),$$

and we have calculated the nonarhimedian component (§5.2) and found it to be

$$\begin{aligned} S_{\Lambda,\infty}((ua)_\infty, (\nu\|a_f\|)_\infty) &= c_1(\nu\|a_f\|)^{3k-1}\ j_k((ua)_\infty, (\nu\|a_f\|)_\infty) \\ &= c_1\nu^{3k-1}\|a_f\|^{3k-1} j_{k,\nu}(ua), \end{aligned}$$

where $j_{k,\nu}(ua)$ is the factor of automorphy appearing in the definition of adelic theta functions. We have also seen that the Siegel function behaves nicely with respect to right translation by elements of $T_{\mathbf{A}}$ (Lemma 5.1) and hence we can simplify the nonarchimedean component using this result:

$$\begin{aligned} S_{\Lambda,f}((ua)_f, (\nu\|a_f\|)_f) &= S_{\Lambda,f}(a_f a_f^{-1} u a_f, (\nu\|a_f\|)_f) \\ &= \Lambda(a_f)\,\|a_f\|^{-(3k-1)}\ S_{\Lambda,f}(a_f^{-1} u a_f, \nu\zeta(a_f)), \end{aligned}$$

where $\zeta : K_f^* \to \mathbf{Z}_f^*$ is the homomorphism defined by $\zeta(a_f) = a_f\overline{a}_f/\mathcal{N}(a_f)$. Combining the above observations, we find that

$$S_\Lambda(ua, \nu\|a_f\|) = c_1\nu^{3k-1}\ j_{k,\nu}(ua)\,\Lambda(a_f)\,S_{\Lambda,f}(a_f^{-1} u a_f, \nu\zeta(a_f)).$$

Now recall that since θ is an adelic theta function of level ν, the function $\tilde{\theta}([w,t]a) = j_{k,\nu}([w,t]a)^{-1}\theta([w,t]a)$ is independent of t_∞ and a_∞. Moreover, the function

$$\theta(w_\infty, u_f a_f) = \tilde{\theta}([w_\infty, 0]u_f a_f),$$

that $\tilde{\theta}$ induces on $N_\infty \backslash M_\infty / T_\infty \cong \mathbf{C}$ is holomorphic.

Combining the observations of the previous two paragraphs we find that if $u = [w, 0]$ then

$$\left(\overline{S_\Lambda([w,0]a, \nu\|a_f\|)\theta}\right)([w,0]a) =$$

$$\left(\overline{c_1\nu^{3k-1}\,\Lambda(a_f)\,S_{\Lambda,f}(a_f^{-1}u_f a_f, \nu\zeta(a_f))}\right)\,|j_k(u_\infty a_\infty, (\nu\|a_f\|)_\infty)|^2\,\theta(w_\infty, u_f a_f)$$

$$= \overline{c_1}\nu^{3k-1}\,h_2(u_f a_f)\,\|a_f\|^{3k}\,|j_k(u_\infty a_\infty, (\nu\|a_f\|)_\infty)|^2\,\theta(w_\infty, u_f a_f).$$

Next we use the definition of j_k to see that

$$\|a_f\|^{3k}\,|j_k(u_\infty a_\infty, (\nu\|a_f\|)_\infty)|^2\,\theta(w_\infty, u_f a_f) =$$

$$\|a_f\|^{3k}\,\|a_\infty\|^{3k}\,e^{-2\pi\nu\sqrt{D}\|a\|}\,e^{-2\pi\|a_f\|\nu\sqrt{D}\|w_\infty\|}\theta(w_\infty, u_f a_f)$$

$$= h_1(a)\,h_3(w_\infty, u_f a_f).$$

Q.E.D.

6.2.3. Application of the residue theorem

Armed with the calculations in the previous section, we can now attack the inner product integral of Lemma 6.2.1. Indeed, applying the previous lemma we find that

$$(E_{\Lambda,\nu}, \theta) = \overline{c_1}\,\nu^{3k-1}\int_{T_{\mathbf{Q}}\backslash T_{\mathbf{A}}} h_1(a)\int_{N_f\backslash U_f} h_2(\dot{u}_f a_f)\int_{W_\infty} h_3(w_\infty, \dot{u}_f a_f)dw_\infty d\dot{u}_f da,$$

where $W = N\backslash U$. To evaluate the inner integral we will apply the residue theorem.

Lemma 6.2.3. *Notation as above,*

$$\int_{W_\infty} h_3(w_\infty, u_f a_f)dw_\infty = c_2\|a_f\|^{-1}\tilde{\theta}(u_f a_f),$$

where

$$c_2 = 2\pi\int_0^\infty e^{-2\pi\nu\sqrt{D}s^2}sds = (2\nu\sqrt{D})^{-1}.$$

Proof: Recall that $W_\infty \cong \mathbf{C}$, and the Haar measure dw_∞ on W_∞ is the standard Haar measure $dxdy$ where $w_\infty = x + iy$. If we change to polar coordinates, $x + iy = re^{it}$, the integral in question becomes

$$\int_{W_\infty} h_3(w_\infty, u_f a_f)dw_\infty = \int_0^\infty\int_0^{2\pi} h_3(se^{it}, u_f a_f)sdsdt$$

$$= \int_0^\infty\int_0^{2\pi}\theta(se^{it}, u_f a_f)e^{-2\pi\nu\sqrt{D}\|a_f\|s^2}dtsds.$$

After making the change of variables $z = e^{it}$, the inner integral becomes a line integral over the circle of radius s in the complex plane:

$$= \int_0^\infty \left(\frac{1}{i} \int_{|z|=s} \frac{\theta(z, u_f a_f)}{z} dz \right) e^{-2\pi\nu\sqrt{D}\|a_f\|s^2} s ds.$$

Since $\theta(w_\infty, m_f)$ is a holomorphic function of w_∞, we can apply the residue theorem to evaluate the inner integral, whose value is seen to be $2\pi\theta(0, u_f a_f)$. The remaining part of the integral is easily evaluated

$$2\pi\theta(0, u_f a_f) \int_0^\infty e^{-2\pi\nu\sqrt{D}\|a_f\|s^2} s ds = 2\pi\theta(0, u_f a_f) \frac{1}{4\pi\nu\sqrt{D}\|a_f\|}$$

$$= \frac{1}{(2\nu\sqrt{D}\|a_f\|} \tilde{\theta}(u_f a_f).$$

Q.E.D.

6.2.4. Completion of the proof of Proposition 6.2

We can now complete the evaluation of the archimedean contribution to the inner product integral $(E_{\Lambda,\nu}, \theta)$, which, after applying the previous lemma, has the form:

$$(E_{\Lambda,\nu}, \theta) = \overline{c_1}\nu^{3k-1} c_2 \int_{T_{\mathbf{Q}}\backslash T_{\mathbf{A}}} h_1(a) \int_{N_f\backslash U_f} h_2(\dot{u}_f a_f)\|a_f\|^{-1}\tilde{\theta}(u_f a_f) d\dot{u}_f da.$$

To do this, we will decompose the integral over $T_{\mathbf{Q}}\backslash T_{\mathbf{A}}$ into a multiple integral by first integrating over T_∞ and then integrating over $T_{\mathbf{Q}}\backslash T_{\mathbf{A}}/T_\infty \cong J_K$. Since $h_1(a)$ is the only term that depends on a_∞, we must evaluate the integral of $h_1(a_\infty a_f)$ over T_∞.

$$\int_{T_\infty} h_1(a_\infty a_f) da_\infty = \int_{\mathbf{C}^*} \|a_\infty a_f\|^{3k} e^{-2\pi\nu\sqrt{D}\|a_\infty a_f\|} da_\infty.$$

Recall that the Haar measure on T_∞ is normalized so that in polar coordinates if $a = re^{2\pi it}$ then $da = drdt$. Since the integrand depends only on the norm of a_∞, we can change coordinates:

$$\int_{T_\infty} h_1(a_\infty a_f) da_\infty = \int_0^1 \int_0^\infty r^{6k} e^{-2\pi\nu\sqrt{D}r^2} \frac{dr}{r} dt,$$

and this evaluates to the constant c_0 (c.f. §3.3.2) defined by

$$c_0 = \frac{(3k-1)!}{2(2\pi\nu\sqrt{D})^{3k}}.$$

Thus, taking this last calculation into account, we can tranform the inner product integral into the form

$$(E_{\Lambda,\nu}, \theta) = \overline{c_1}\nu^{3k-1} c_2 c_0 \int_{J_K} \int_{W_f} h_2(w_f a_f)\|a_f\|^{-1}\tilde{\theta}(w_f a_f) dw_f da_f,$$

where da_f is the quotient measure on $J_K = T_{\mathbf{Q}}T_\infty \cap T_f \backslash T_f$. Thus, if we observe that $\epsilon_\infty = \overline{c_1}\nu^{3k-1}c_2c_0$, and if we substitute in the value of h_2

$$h_2(u_f a_f) = \|a_f\|^{-3k}\, \overline{\Lambda(a_f)\ S_{\Lambda,f}(a_f^{-1}u_f a_f, \nu a_f \overline{a}_f/\mathcal{N}(a_f))},$$

into this formula, we obtain

$$= \epsilon_\infty \int_{J_K}\int_{W_f} \|a_f\|^{-3k}\ \|a_f\|^{-1}\ \overline{\Lambda(a_f)S_{\Lambda,f}(a_f^{-1}w_f a_f, \nu a_f\overline{a}_f/\mathcal{N}((a_f)))}\tilde{\theta}(w_f a_f)dw_f da.$$

Making the change of variables $v_f = a_f^{-1}w_f a_f$, with $dv_f = \|a_f\|^{-1}dw_f$ yields the integral

$$= \epsilon_\infty \int_{J_K}\int_{W_f} \|a_f\|^{-3k}\ \overline{\Lambda(a_f)S_{\Lambda,f}(v_f, \nu\zeta(a_f))}\tilde{\theta}(a_f v_f)dv_f da.$$

Since $\Lambda(x\overline{x}) = \|x\|^{3k}$ (§1.6), we see that $\overline{\Lambda(x)}\|x\|^{-3k} = \Lambda(x^{-1})$ and so we have proved that

$$(E_{\Lambda,\nu}, \theta) = \epsilon_\infty \int_{J_K}\int_{W_f} \mathcal{S}_{\Lambda,\theta}(a, w)dwda,$$

and so to complete the proof of Proposition 6.2, we need only show that $\mathcal{S}_{\Lambda,\theta}(a, w)$ is defined on the product space $J_K \times W_f$. This is the content of the next lemma.

Lemma 6.2.4. *Let $x \in T_{\mathbf{Q}}$; then for all $a \in T_f$ and $w \in W_f$, we have*

$$\mathcal{S}_{\Lambda,\theta}(x_f a, w) = \mathcal{S}_{\Lambda,\theta}(a, w).$$

Proof: This follows from the following easily verified facts:

$$\Lambda((x_f a_f)^{-1}) = \Lambda(x_\infty)\Lambda(a_f^{-1}) = jac(x_\infty, o)^k \Lambda(a_f^{-1}) \qquad \text{(c.f. §1.7)}$$

$$\tilde{\theta}(x_f w_f a_f) = jac(x_\infty, o)^{-k}\tilde{\theta}(w_f a_f) \qquad \text{(c.f. Lemma 3.1.1)}$$

and finally since $\mathcal{N}(x_f) = x\overline{x}$ for all $x \in K^*$, we see that $\zeta(x_f a_f) = \zeta(a_f)$ and so

$$S_{\Lambda,f}(a_f^{-1}w_f a_f, \nu\|x_f a_f\|) = S_{\Lambda,f}(a_f^{-1}w_f a_f, \nu\|a_f\|).$$

These three observations prove the lemma. **Q.E.D.**

6.3. The contribution of the unramified nonarchimedean places

In this section, we evaluate the contribution of the nonarchimedean places not dividing νD to the inner product formula

$$(E_{\Lambda,\nu}, \theta) = \epsilon_\infty \int_{J_K \times W_f} \mathcal{S}_{\Lambda,\theta}(\beta, w)dwd\beta,$$

which was established in the previous section (Prop. 6.2), where $\mathcal{S}_{\Lambda,\theta}$ is the function on $J_K \times W_f$ defined by

$$\mathcal{S}_{\Lambda,\theta}(\beta, w) = \Lambda(\beta)^{-1}\overline{S_{\Lambda,f}(w, \nu\zeta(\beta))}\tilde{\theta}(aw).$$

We will prove the following Proposition:

Proposition 6.3. *Let Λ be a Hecke character of weight k on the maximal torus $D_{\mathbf{A}}$ of $G_{\mathbf{A}}$, let E_Λ be the associated Eisenstein series on $G_{\mathbf{A}}$, let ν be a positive integer, and let $W_{\nu D} = \prod_{p|\nu D} W_p$. Then for any $\xi \in \Xi_\nu$ and any $\theta \in V_\xi$ (c.f. §4.3)*

$$(E_{\Lambda,\nu}, \theta) = \epsilon_\infty \left(\prod_{p \nmid \nu D} \frac{\epsilon_p(k,\Lambda,\xi)}{1-p^{-3k}} \right) \int_{J_K \times W_{\nu D}} \mathcal{S}_{\Lambda,\theta}(\beta, w) dw d\beta,$$

where the constant ϵ_∞, the function $\mathcal{S}_{\Lambda,\theta}$, and the idele class group J_K are as in Proposition 6.2; and the constant $\epsilon_p(k,\Lambda,\xi)$ is the local L-factor of the L-series $\overline{L_\xi(k,\Lambda,\xi)}$ given in §6.1.4. If p is inert then $\epsilon_p(k,\Lambda,\xi) = 1-p^{-3k}$ and if p splits in K, $p = \pi\overline{\pi}$, and $\xi = (\nu, \kappa^*, \Sigma)$, then

$$\epsilon_p(k,\Lambda,\xi) = \frac{1-p^{-(3k-1)}}{1} \frac{1-\overline{\chi}_1^*(\pi)p^{-3k/2}}{1-\chi_1^*\overline{\chi}_2^*(\pi)p^{-(3k-1)/2}} \frac{1-\overline{\chi}_1^*(\overline{\pi})p^{-3k/2}}{1-\chi_1^*\overline{\chi}_2^*(\overline{\pi})p^{-(3k-1)/2}},$$

where $\chi_1^*(x) = \Lambda_Z^2 \Lambda_T^{*3}(x)$ and $\chi_2^*(x) = \Lambda_T^{*2}\kappa^*(x)$.

6.3.1. Reduction to the local case

Since the integrand does not factor into a product of local functions, we cannot simply express this integral as a product of integrals but must use a limiting argument instead. More precisely, we will show that if p is a prime which does not divide νD, and if $v \in W_f$ is an adele whose p-component v_p is zero, then for any $\xi \in \Xi_\nu$ and any $\theta \in V_\xi$, we have

$$\int_{J_K \times W_p} \mathcal{S}_{\Lambda,\theta}(a, v+w_p) dw_p da = \frac{\epsilon_p(k,\Lambda,\xi)}{1-p^{-3k}} \int_{J_K} \mathcal{S}_{\Lambda,\theta}(a,v) da,$$

where $\epsilon_p(k,\Lambda,\xi)$ is the local factor of the L-series $L_\xi(k,\Lambda)$ given explicitly in §6.1.4. Since p does not divide νD, it is either inert or split and the proof will proceed in two cases accordingly. If p remains inert in K, then the proof will follow immediately from the fact that the local Siegel function is supported on O_p in this case. The case when p splits in K is more interesting since the local Siegel function is not compactly supported on K_p.

First we show how to reduce the problem to one of evaluating local integrals. For any set S of primes, let W_S be the closed subgroup of W_f consisting of those adeles whose p-components are 0 for all p not in S, and let $dw_S = \prod_{p \in S} dw_p$ be the Haar measure on W_S. If S' is the set of primes not in S, then $W_f \cong W_S \times W_{S'}$ and if g is an L^1 function on W_f whose restriction to almost any coset of W_S induces an L^1 function of $W_{S'}$, then

$$\int_{W_f} g(w) dw = \int_{W_{S'}} \int_{W_S} g(w_S + w_{S'}) dw_S dw_{S'}.$$

Lemma 6.3.1. *To prove Proposition 6.3, it will suffice to show that if p is a prime not dividing νD, S is a set of primes not containing p and $v \in W_S$ is fixed, then*

$$\int_{J_K \times W_p} \mathcal{S}_{\Lambda,\theta}(\beta, v+w_p) dw_p d\beta = \frac{\epsilon_p(k,\Lambda,\xi)}{1-p^{-3k}} \int_{J_K} \mathcal{S}_{\Lambda,\theta}(\beta, v) d\beta.$$

Proof: By induction, the equality in the lemma implies that for any finite set of primes S which contains S_0, one has

$$\int_{J_K\times W_S}\mathcal{S}_{\Lambda,\theta}(\beta,v)dvd\beta=\left(\prod_{p\in S-S_0}\frac{\epsilon_p(k,\Lambda,\xi)}{1-p^{-3k}}\right)\int_{J_K\times W_{S_0}}\mathcal{S}_{\Lambda,\theta}(\beta,w_0)dw_0d\beta.$$

Since the integrand is left $W(L)_f$ invariant, we can rewrite this equation as an equality among integrals on an open subset $J_K\times V_S$ of $J_K\times W_f$ where $V_S=W_S\times\prod_{p\notin S}W(L)_p$, is an open set and every compact subset of W_f is contained in some V_S.

$$\int_{J_K\times V_S}\mathcal{S}_{\Lambda,\theta}(\beta,w)dwd\beta=\left(\prod_{p\in(S-S_0)}\frac{\epsilon_p(k,\Lambda,\xi)}{1-p^{-3k}}\right)\int_{J_K\times V_{S_0}}\mathcal{S}_{\Lambda,\theta}(\beta,w)dwd\beta.$$

Since $\mathcal{S}_{\Lambda,\theta}\in L_1(J_K\times W_f)$, the limit of the integrals over $J_K\times V_S$ as S varies, exists and is equal to the integral over W_f. (c.f. [8, Ch.XIV, §5]) **Q.E.D.**

The evaluation of the integral in the previous lemma splits naturally into two cases, depending on whether p is inert in K or splits in K.

6.3.2. The case of p inert in K

Lemma 6.3.2. *Notation as in Proposition 6.3. Let p be inert in K, and let $\epsilon_p(k,\Lambda,\xi)=1-p^{-3k}$ be the p-factor of the L-series $L_\xi(k,\Lambda)$ (c.f. §6.1.4). Then for every fixed $\beta\in T_f$ and every fixed $v\in W_f$ such that $v_p=0$, we have*

$$\int_{W_p}\mathcal{S}_{\Lambda,\theta}(\beta,v+w_p)dw_p=\frac{\epsilon_p(k,\Lambda,\xi)}{1-p^{-3k}}\mathcal{S}_{\Lambda,\theta}(\beta,v).$$

Proof: To prove this lemma, it will suffice to show for all $w_p\in W_p$, $v\in W_f$ with $v_p=0$ and $\beta\in T_f$ that

$$\mathcal{S}_{\Lambda,\theta}(\beta,v+w_p)=\begin{cases}\mathcal{S}_{\Lambda,\theta}(\beta,v) & \text{if } w_p\in O_p,\\ 0 & \text{otherwise .}\end{cases}$$

since $W(L)_p$ has measure 1. Recall that $\mathcal{S}_{\Lambda,\theta}$ is the function on $T_f\times W_f$ defined by

$$\mathcal{S}_{\Lambda,\theta}(\beta,w)=\Lambda_T(\beta)^{-1}\overline{S_{\Lambda,f}([w,t_w],\nu\zeta(\beta))}\tilde{\theta}(\beta[w,t_w]),$$

for any t_w in $\mathbf{Q}_f$ Since the Siegel function $S_{\Lambda,f}$ and the theta function θ are left $U(L)_f$ invariant, $\mathcal{S}_{\Lambda,\theta}$ is left $W(L)_f$ invariant. Recall that the Siegel function factors into local Siegel functions, so

$$S_{\Lambda,f}([w,t],\beta)=\prod_p S_{\Lambda,p}([w_p,t_p],\beta_p),$$

and we can use Proposition 5.4.5 to evaluate these local Siegel functions. By assumption, ν is a unit in $\mathbf{Z}_p$, so $(\nu\zeta(\beta))_p$ is in $\mathbf{Z}_p^*$ and so Proposition 5.4.5 asserts that

$$S_{\Lambda,p}([w_p,-w_p\overline{w}_pD/2],\nu\zeta(\beta))=\begin{cases}1-p^{-3k} & \text{if } w_p\in O_p,\\ 0 & \text{otherwise.}\end{cases}$$

So $S_{\Lambda,p}([w_p, -w_p\overline{w}_p D/2], \nu\zeta(\beta)) = S_{\Lambda,p}([0,0], \nu\zeta(\beta))$ and this implies that $\mathcal{S}_{\Lambda,\theta}(\beta, v+w_p)$ is zero unless $w_p \in O_p$ in which case it equals $\mathcal{S}_{\Lambda,\theta}(\beta, v)$. **Q.E.D.**

6.3.3. First step for the case of p split in K

Our strategy is to decompose the integral into a sum of integrals over $J_K \times C_{a,b}$ where $C_{a,b} = \pi^{-a}\overline{\pi}^{-b}O_p^* + O_p$ where π is a prime of K dividing p. The local Siegel function admits a relatively simple expression over these domains, and is nonzero only on the domains $C_{a,0}$ and $C_{0,b}$ for $a, b \geq 0$. Since the integrals over the domains $C_{a,0}$ and $C_{0,b}$ are symmetric (depending on the choice of prime π dividing p) we need only treat the integral over $C_{a,0}$ in detail. Our next step is to relate the integral

$$\int_{C_{a,0}} \mathcal{S}_{\Lambda,\theta}(a, v+w_p)dw_p,$$

to the action of the Shintani operator $l(y_\pi^a)$ discussed in Chapter 4, where $y_\pi = \pi/\overline{\pi}$. By making this relation explicit we show that there are constants α_i, $i = 1, 2, 3$ such that

$$\int_{C_{a,0}} \mathcal{S}_{\Lambda,\theta}(a, v+w_p)dw_p = \alpha_1\alpha_2^a\alpha_3^a\,\mathcal{S}_{\Lambda,\theta}(ay_\pi^a, v+w_p) - \alpha_1\alpha_2^a\alpha_3^{a-1}\mathcal{S}_{\Lambda,\theta}(ay_\pi^{a-1}, v+w_p).$$

Next we use the translation invariance of the Haar measure on J_K to show that

$$\int_{J_K \times C_{a,b}} \mathcal{S}_{\Lambda,\theta}(a, v+w_p)dw_p da = \sigma_{a,b}\int_{J_K} \mathcal{S}_{\Lambda,\theta}(a, v)da,$$

where $\sigma_{0,0} = 1$, $\sigma_{a,b} = \overline{\sigma_{b,a}}$, $\sigma_{a,b} = 0$ unless one of a,b is zero, and if $a > 0$,

$$\sigma_{a,0} = \alpha_1\alpha_2^a\alpha_3^a(1-\alpha_3^{-1}).$$

The constants α_i involve the values of eigencharacter κ^* of θ and the values of the Hecke character Λ used to define the Eisenstein series.

In this subsection we evaluate the integral of $\mathcal{S}_{\Lambda,\theta}(\beta, v)$ over $C_{a,b}$, in the next subsection we evaluate the integral over $J_K \times C_{a,b}$, and in the third subsection we put the results together to obtain a formula for the integral over $J_K \times W_p$.

First, we recall some notation concerning the completion $K_p = K \otimes \mathbf{Q}_p$ of K, where p is a prime that splits in K (c.f. §5.5.1). Let e, $\overline{e}$ be the idempotents in K_p as in §5.5.1, so that $K_p = \mathbf{Q}_p e \oplus \mathbf{Q}_p \overline{e}$ and $e + \overline{e} = 1$, $e\overline{e} = 0$. Moreover there is a $\tau_1 \in \mathbf{Q}_p$ such that $\sqrt{-D} = \tau_1 e + \tau_1 \overline{e}$. The prime ideals of O_p are generated by π and $\overline{\pi}$ where we may take $\pi = pe + \overline{e}$. Recall that $W_p \cong K_p$ and $W(L)_p \cong O_p$ under the isomorphism that takes w to the N_p coset of $[w, w\overline{w}\tau_1/2]$.

Let $C_{a,b} = O_p^*\pi^{-a}\overline{\pi}^{-b} + O_p$, then K_p is the union of the sets $C_{a,b}$ where $a, b \geq 0$, so the integral of $\mathcal{S}_{\Lambda,\theta}(\beta, v+w_p)$ over W_p can be decomposed into a sum of the integrals over these sets $C_{a,b}$ for all $a, b \geq 0$:

$$\int_{J_K}\int_{W_p} \mathcal{S}_{\Lambda,\theta}(\beta, v+w_p)dw_p d\beta = \sum_{a,b\geq 0}\int_{J_K}\int_{C_{a,b}} \mathcal{S}_{\Lambda,\theta}(\beta, v+w_p)dw_p d\beta.$$

First we evaluate the integral over $C_{a,b}$.

Lemma 6.3.3. *Let $\mathcal{S}_{\Lambda,\theta}$ be the function on $J_K \times W_f$ defined in Proposition 6.2, let χ_1, χ_2 be the ideal group characters defined in §6.1.4, let a, b be positive integers, and let $y_\pi = \pi/\overline{\pi}$. Then the integral of $\mathcal{S}_{\Lambda,\theta}$ over the subsets $C_{a,b} = \pi^a\overline{\pi}^b O_p^* + O_p$ is given as follows:* $\int_{C_{a,b}} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p =$

$$\begin{cases} \mathcal{S}_{\Lambda,\theta}(\beta, v) & \text{if } a = b = 0, \\ 0 & \text{if } a, b > 0, \\ \overline{\alpha_1\alpha_2^a\alpha_3^a}\left(\mathcal{S}_{\Lambda,\theta}(\beta y_\pi^{\frac{a}{\pi}}, v) - \overline{\alpha_2^{-1}}\mathcal{S}_{\Lambda,\theta}(\beta y_{\overline{\pi}}^{a-1}, v)\right) & \text{if } a = 0,\ b > 0\ , \\ \alpha_1\alpha_2^a\alpha_3^a\left(\mathcal{S}_{\Lambda,\theta}(\beta y_\pi^a, v) - \alpha_2^{-1}\mathcal{S}_{\Lambda,\theta}(\beta y_\pi^{a-1}, v)\right) & \text{if } a > 0,\ b = 0, \end{cases}$$

where $\alpha_1 = (1 - \overline{\chi}_1(\pi)) / (1 - p^{-3k})$, $\alpha_2 = \chi_1(\pi)$, and $\alpha_3 = \overline{\chi}_2(\pi)$.

Proof: The first two assertion follow easily from the formula for the local Siegel function. Indeed, recall that the integrand is given by

$$\mathcal{S}_{\Lambda,\theta}(\beta, w) = \Lambda_T(\beta)^{-1}\ \overline{S_{\Lambda,f}([w, t_w], \nu\zeta(\beta))}\ \tilde{\theta}(\beta[w, t_w]),$$

where $\zeta(\beta) = \beta\overline{\beta}/\mathcal{N}(\beta)$ and t_w is any element of $\mathbf{Q}_f$.

Since $\tilde{\theta}$ and $S_{\Lambda,f}([w, t_w], \nu\zeta(\beta))$ are $W(L)_p$ invariant, the integrand $\mathcal{S}_{\Lambda,\theta}(\beta, v + w_p)$ is also. Therefore, since the measure of $W(L)_p$ is 1, the integral of $\mathcal{S}_{\Lambda,\theta}(\beta, v + w_p)$ over $C_{0,0} = W(L)_p$ is $\mathcal{S}_{\Lambda,\theta}(\beta, v)$ as claimed.

Observe now that $\nu\zeta(\beta)$ is a unit in $\mathbf{Z}_p^*$, since by assumption p does not divide ν. Thus, we can apply Corollary 5.5.6, which provides a formula for the value of the local Siegel function at a point in $C_{a,b}$ in terms of a, b. We recall the result:

i) if $w_p \in C_{0,0} = O_p$, then $S_{\Lambda,p}([w_p, -w_p\overline{w}_p\tau_1/2], \nu\zeta(\beta)) = 1 - p^{-3k}$
ii) if $w_p \in C_{a,b}$ with $a, b > 0$, then $S_{\Lambda,p}([w_p, -w_p\overline{w}_p\tau_1/2], \nu\zeta(\beta)) = 0$
iii) if $w_p \in C_{a,0}$ with $a > 0$, then $S_{\Lambda,p}([w_p, -w_p\overline{w}_p\tau_1/2], \nu\zeta(\beta)) = \overline{\alpha_1\alpha_2^a}$
iv) if $w_p \in C_{0,b}$ with $b > 0$, then $S_{\Lambda,p}([w_p, w_p\overline{w}_p\tau_1/2], \nu\zeta(\beta)) = \alpha_1\alpha_2^a$

where α_1, α_2 are as in the statement of the lemma.

So, let a, b be two positive integers, and observe that

$$\overline{S_p([w_p, 0], \nu\zeta(\beta))} = 0,$$

for $w \in C_{a,b}$ which implies that the integral of $\mathcal{S}_{\Lambda,\theta}(\beta, v + w_p)$ over $C_{a,b}$ is zero.

Thus, we need only evaluate the integrals over $C_{a,0}$ and $C_{0,b}$ for $a, b > 0$, and by exploiting symmetry we will only need to consider the integral over $C_{a,0}$ in depth.

By applying the explicit formula for the value of the local Siegel function in this case we obtain the following formula for the value of the nonarchimedean part of the Siegel function when $w_p \in C_{a,0}$

$$\overline{S_{\Lambda,f}([v + w_p, -w_p\overline{w}_p\tau_1/2], \nu\zeta(\beta))} = \overline{\frac{\overline{\alpha_1\alpha_2^a}}{1 - p^{-3k}}}\,\overline{S_{\Lambda,f}([v, 0], \nu\zeta(\beta))}.$$

If we now use this formula in the integral we are trying to evaluate, we find that

$$\int_{C_{a,0}} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p =$$

$$\alpha_1\alpha_2^a\ \Lambda(\beta)^{-1}\ \overline{S_{\Lambda,f}([v,0],\nu\zeta(\beta))}\ \int_{C_{a,0}} \tilde{\theta}(\beta[v+w_p,-w_p\overline{w}_p\tau_1/,/2])\,dw_p.$$

We will now use the assumption that $\theta \in V_\xi$ to evaluate the inner integral in the preceeding equation.

Claim. *Let $\theta \in V_\xi$ for $\xi = (\nu,\kappa^*,\Sigma)$ and let $\beta \in T_f$, $v \in W_f$ with $v_p = 0$, and let $a > 0$. Then*

$$\int_{C_{a,0}} \tilde{\theta}(\beta[v+w_p,-w_p\overline{w}_p\tau_1/,/2])\,dw_p =$$

$$\left(p^{1/2}\,\overline{\kappa}^*(\pi)\right)^a \tilde{\theta}(\beta y_\pi^a[v,0]) - \left(p^{1/2}\,\overline{\kappa}^*(\pi)\right)^{a-1} \tilde{\theta}(\beta y_\pi^{a-1}[v,0]).$$

Proof of Claim. First we express the integral as an integral over $p^{-a}\mathbf{Z}_p^*e \subset K_p$. Observe that if $w_p = xe + y\overline{e}$ for $x, y \in \mathbf{Q}_p$, then

$$[w_p, -w_p\overline{w}_p\tau_1/2] = [xe,0],[y\overline{e},0].$$

Since $w_p \in C_{a,0}$, we have $[y\overline{e},0] \in U(L)_p$, so the integral in the claim can be replaced by an integral over $p^{-a}\mathbf{Z}_p^*e$:

$$\int_{C_{a,0}} \tilde{\theta}(\beta[v+w_p,-w_p\overline{w}_p\tau_1/2])dw_p = \int_{p^{-a}\mathbf{Z}_p^*e} \tilde{\theta}(\beta[v,0][w_p,0])dw_p.$$

We can now use the assumption that θ is an eigenfunction of Shintani's representation to evaluate this last integral. Since $\theta \in V_{k,\nu}(L)_\kappa$, it satisfies (by definition) the following equation for $y_\pi = \overline{\pi}/\pi$ (c.f. §4.2)

$(l_1(y_\pi^a)\theta)\,(\beta[v,0]) =$

$$N(num(y_\pi^a))^{1/2} \int_{U(L)_p} \tilde{\theta}(\beta[v,0]uy_\pi^{-a})du = \kappa(y_\pi^a)\,\tilde{\theta}(\beta[v,0]),$$

and $\kappa(y_\pi^a) = \overline{\kappa}^*(\pi)^a$. We will now show that the integral in the middle can be transformed into an integral over $\{[w,0] : w \in p^{-a}\mathbf{Z}_pe\} \subset U_p$, and this will allow us to evaluate the integral of $\tilde{\theta}$ over $p^{-a}\mathbf{Z}_p^*$. First, we make the change of variables $u' = y_\pi^a u y_\pi^{-a}$ which has modulus 1, and we find

$$\int_{U(L)_p} \tilde{\theta}(\beta[v,0]uy_\pi^{-a})du = \int_{U(y_\pi^a L)_p} \tilde{\theta}(\beta[v,0]y_\pi^{-a}u')du'.$$

Since $U(L)_p = \{[w, t - w\overline{w}\tau_1/2] : w \in O_p, t \in \mathbf{Z}_p\}$ we see that

$$U(L)_p = \{[xe,0][y\overline{e},0][0,t] : x,y,t \in \mathbf{Z}_p\},$$

and hence that $U(y_\pi^a L)_p = \{[p^{-a}xe,0][p^a y\overline{e},0][0,t] : x,y,t \in \mathbf{Z}_p\}$. Since $\tilde{\theta}$ is left $U(L)_p$ invariant, we see that

$$\tilde{\theta}(\beta[v,0]y_\pi^{-a}[p^{-a}xe,0][p^a y\overline{e},0][0,t]) = \tilde{\theta}(\beta[v,0]y_\pi^{-a}[xe,0]),$$

for all $x, y, t \in \mathbf{Z}_p$, and so

$$\int_{U(L)_p} \tilde{\theta}(\beta[v,0]uy_\pi^{-a})du = \int_{p^{-a}\mathbf{Z}_p e} \int_{p^a \mathbf{Z}_p \bar{e}} \int_{\mathbf{Z}_p} \tilde{\theta}(\beta[v,0]y_\pi^{-a}u)du$$

$$= p^{-a} \int_{p^{-a}\mathbf{Z}_p e} \tilde{\theta}(\beta[v,0]y_\pi^{-a}u)du.$$

This shows that the fact that θ is an eigenfunction of the Shintani operator $l(y_\pi^a)$ is equivalent to the following property

$$\int_{p^{-a}\mathbf{Z}_p e} \tilde{\theta}(\beta y_\pi^{-a}[v,0]u)du = \int_{U(L)_p} \tilde{\theta}(\beta[v,0]uy_\pi^{-a})du = p^{a/2}\,\overline{\kappa}^*(\pi)^a \tilde{\theta}(\beta[v,0]),$$

and if we replace β by βy_π^a we obtain

$$\int_{p^{-a}\mathbf{Z}_p e} \tilde{\theta}(\beta[v,0]u)du = \left(p^{1/2}\,\overline{\kappa}^*(\pi)\right)^a \tilde{\theta}(\beta y_\pi^a[v,0]).$$

We are actually interested in the integral over $p^{-a}\mathbf{Z}_p^*$ which is easily seen to be

$$\int_{p^{-a}\mathbf{Z}_p^* e} \tilde{\theta}(\beta[v,0]u)du = \left(p^{1/2}\,\overline{\kappa}^*(\pi)\right)^a \tilde{\theta}(\beta y_\pi^a[v,0]) - \left(p^{1/2}\,\overline{\kappa}^*(\pi)\right)^{a-1} \tilde{\theta}(\beta y_\pi^{a-1}[v,0]).$$

This proves the claim.

Thus, using the Claim, we have obtained the following partial evaluation of the integral in the Lemma:

$$\int_{C_{a,0}} \mathcal{S}_{\Lambda,\theta}(\beta, v+w_p)dw_p = \alpha_1\alpha_2^a\; p^{a/2}\,\overline{\kappa}^*(\pi)^a \Lambda(\beta)^{-1}\left(I(\beta y_\pi^a, v) - I(\beta y_\pi^{a-1}, v)\right),$$

where

$$I(\beta, v) = \overline{S_{\Lambda,f}([v,0], \nu\zeta(\beta))}\,\tilde{\theta}(\beta y_\pi^a[v,0]),$$

and we have used the fact that $\zeta(y_\pi) = 1$ and ζ is a homomorphism. Thus, if we let

$$\alpha_3 = \Lambda(y_\pi)^a p^{a/2}\,\overline{\kappa}^*(\pi)^a,$$

we see that $\alpha_3 = \overline{\chi}_2(\pi)$, where $\overline{\chi}_2(z) = \overline{\Lambda_T^{*2}\kappa^*}(z)\|z\|^{-1/2}$ is the character on the group of ideals prime to νD defined in §6.1.4. Moreover, since $\mathcal{S}_{\Lambda,\theta}(\beta, w) = \Lambda(\beta)^{-1}I(\beta, w)$, we see that

$$\int_{C_{a,0}} \mathcal{S}_{\Lambda,\theta}(\beta, v+w_p)dw_p = \alpha_1\alpha_2^a\;\alpha_3^a\left(\mathcal{S}_{\Lambda,\theta}(\beta y_\pi^a, v) - \alpha_3^{-1}\mathcal{S}_{\Lambda,\theta}(\beta y_\pi^{a-1}, v)\right),$$

as was to be shown. The proof in the case of $C_{0,b}$ is completely symmetric to this case. **Q.E.D.**

6.3.4. Integrating over J_K

Next we integrate over the domain $J_K \times C_{a,b}$ and use the translation invariance of the Haar measure on J_K to simplify the result of integrating over $C_{a,b}$.

Lemma 6.3.4. *Let $\mathcal{S}_{\Lambda,\theta}$ be the function on $J_K \times W_f$ defined in Proposition 6.2, let χ_1, χ_2 be the characters on the group of ideals prime to νD, defined in §6.1.4, and let $C_{a,b} = \pi^a \overline{\pi}^b O_p^* + O_p$ for a split prime $p = \pi\overline{\pi}$. Then*

$$\int_{J_K} \int_{C_{a,b}} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p d\beta = \sigma_{a,b} \int_{J_K} \mathcal{S}_{\Lambda,\theta}(\beta, v) d\beta,$$

where $\sigma_{0,0} = 1$, $\sigma_{a,b} = \overline{\sigma_{b,a}}$, $\sigma_{a,b} = 0$ unless one of a and b is zero, and if $a > 0$

$$\sigma_{a,0} = \alpha_1 \left(\alpha_2 \alpha_3\right)^a \left(1 - \alpha_3^{-1}\right),$$

where $\alpha_1 = \left(1 - \overline{\chi}_1(\pi)\right) / \left(1 - p^{-3k}\right)$, $\alpha_2 = \chi_1(\pi)$, and $\alpha_3 = \overline{\chi}_2(\pi)$.

Proof: Applying the previous Lemma, we find that the integral in question can be written as follows:

$\int_{J_K} \int_{W_p} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p d\beta =$

$$= \alpha_1 \alpha_2^a \alpha_3^a \int_{J_K} \mathcal{S}_{\Lambda,\theta}(\beta y_\pi^a, v + w_p) d\beta - \int_{J_K} \alpha_1 \alpha_2^a \alpha_3^{a-1} \mathcal{S}_{\Lambda,\theta}(\beta y_\pi^{a-1}, v + w_p) d\beta,$$

and by the translation invariance of the Haar measure $d\beta$, we see that $\int_{J_K} \int_{W_p} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p d\beta$ is equal to

$$\alpha_1 \alpha_2^a \alpha_3^a (1 - \alpha_3^{-1}) \int_{J_K} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) d\beta.$$

Similarly, $\sigma_{0,b} = \overline{\sigma}_{b,0}$. **Q.E.D.**

6.3.5. Completion of the case of p split and prime to νD

Returning to the original problem of evaluating the integral over W_p, we need only evaluate the sum of the constants $\sigma_{a,b}$, over all a, b.

Lemma 6.3.5. *Notation as in Proposition 6.3. Then*

$$\int_{J_K} \int_{W_p} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p d\beta = \frac{\epsilon_p(k, \Lambda, \xi)}{1 - p^{-3k}} \int_{J_K} \mathcal{S}_{\Lambda,\theta}(\beta, v) d\beta.$$

Proof: By the previous lemma, we know that

$$\int_{J_K} \int_{W_p} \mathcal{S}_{\Lambda,\theta}(\beta, v + w_p) dw_p d\beta = \epsilon_p' \int_{J_K} \mathcal{S}_{\Lambda,\theta}(\beta, v) d\beta,$$

where, with the notation $\mathcal{T}(z) = z + \overline{z}$, we have $\epsilon_p' = 1 + \sum_{a=1}^{\infty} \mathcal{T}(\sigma_{a,0})$, If we let $\chi_1^*(z) = \Lambda_Z^2 \Lambda_T^{*3}(z)$ and $\chi_2^*(z) = \Lambda_T^{*2} \kappa^*(z)$, be the unitary characters associated to χ_1, $\overline{\chi}_2$, then

$\chi_1(x) = \chi_1^*(x)\|x\|^{3k/2}$ and $\chi_2(x) = \chi_2^*(x)\|x\|^{-1/2}$, and after some simplification, we find that

$$\sigma_{a,0} = \frac{\left(1-\overline{\chi}_1^*(\pi)p^{-3k/2}\right)}{1-p^{-3k}} (\chi_1^*(\pi)p^{-3k/2})^a (p^{1/2}\overline{\Lambda}_T^{*2}\overline{\kappa}^*(\pi))^a \left(1-p^{1/2}\overline{\Lambda}_T^{*2}\overline{\kappa}^*(\pi))^{-1}\right)$$

$$= \frac{1-\overline{\chi}_1^*(\pi)p^{-3k/2}}{1-p^{-3k}} \left(1-\chi_2^*(\pi)p^{-1/2}\right)\left(\chi_1^*\overline{\chi}_2(\pi)p^{-(3k-1)/2}\right)^a.$$

Therefore, we can continue the evaluation of $\epsilon'_p = 1 + \sum_{a=1}^{\infty} \mathcal{T}(\sigma_{a,0}) =$,

$$1+\mathcal{T}\left(\sum_{a>0} \frac{1-\overline{\chi}_1^*(\pi)p^{-3k/2}}{1-p^{-3k}} \left(1-\chi_2^*(\pi)p^{-1/2}\right)\left(\chi_1^*\overline{\chi}_2(\pi)p^{-(3k-1)/2}\right)^a\right).$$

After evaluating the summation inside the $\mathcal{T}$ this becomes

$$1+\mathcal{T}\left(\frac{1-\overline{\chi}_1^*(\pi)p^{-3k/2}}{1-p^{-3k}} \left(1-\chi_2^*(\pi)p^{-1/2}\right) \frac{\chi_1^*\overline{\chi}_2^*(\pi)p^{-(3k-1)/2}}{1-\chi_1^*\overline{\chi}_2^*(\pi)p^{-(3k-1)/2}}\right).$$

A straightforward calculation, using only the fact that the values of $\chi_1^*(\pi)$ and $\chi_2^*(\pi)$ have absolute value 1, shows that this expression is precisely equal to $\epsilon_p(k,\Lambda,\xi)/(1-p^{-3k})$, that is

$$\epsilon'_p = \frac{1-p^{-(3k-1)}}{1-p^{-3k}} \frac{1-\overline{\chi}_1^*(\pi)p^{-3k/2}}{1-\chi_1^*\overline{\chi}_2^*(\pi)p^{-(3k-1)/2}} \frac{1-\overline{\chi}_1^*(\overline{\pi})p^{-3k/2}}{1-\chi_1^*\overline{\chi}_2^*(\overline{\pi})p^{-(3k-1)/2}}$$

$$= (1-p^{-3k})^{-1} \frac{1}{\epsilon_{K,p}(3k/2,\Lambda_Z^2\Lambda_T^{*3})} \frac{\epsilon_{K,p}((3k-1)/2,\Lambda_Z^2\Lambda_T^*\overline{\kappa}^*)}{\epsilon_{\mathbf{Q},p}(3k-1,\chi_K))}$$

$$= \epsilon_p(k;\Lambda,\xi).$$

Q.E.D.
This completes the proof of Proposition 6.3.

6.4. The contribution of the ramified nonarchimedean places

We now assume that $\theta \in V_{k,\nu}(L)^0_{\kappa,\Sigma}$ for some set Σ of primes dividing D, and will complete the evaluation of (E_Λ,θ) by determining the contribution of the primes dividing νD, which are the primes dividing the conductor of the character κ. To evaluate the contributions of the ramified primes we will need to assume that the theta function is not only an eigenfunction of Shintani's representation, but is also an eigenfunction of the projections associated to the ramified primes. Similarly, to evaluate the contributions of the unramified primes that divide ν, we will need to assume that θ is primitive. These assumptions will allow us to use the same techniques as in the previous section. The result of these calculations is summarized in the following proposition, which will complete the evaluation of the primitive component of $E_{\Lambda,\nu}$.

Proposition 6.4. *Let $\theta \in V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$, let $\mathcal{S}_{\Lambda,\theta}$ be the function defined in §6.2, and let $W_{\nu D} = \prod_{p|\nu D} W_p$. Then*
$\left(\prod_{p \not| \nu D}(1-p^{-3k})^{-1}\right) \int_{J_K \times W_{\nu D}} \mathcal{S}_{\Lambda,\theta}(\beta, w) dw d\beta =$

$$\left(\prod_{p|\nu D} \epsilon_p(k,\Lambda,\xi)\right) \int_{J_K} \Lambda(\beta)^{-1}\tilde{\theta}(\beta)d\beta,$$

where the constants $\epsilon_p(k,\Lambda,\xi)$ are the local factors of the L-series $\overline{L_\xi(k,\Lambda,\xi)}$ (c.f. §6.1.4).

6.4.1. Reduction to the local case

To prove this proposition we will first show that it suffices to perform certain local calculations, as in the proof of the proposition. The local calculations divide naturally into three cases, according as the prime splits, ramifies, or remains inert in K. Thus, the proposition will be proved in a sequence of four lemmas.

Lemma 6.4.1. *To prove Proposition 6.4, it will suffice to show for any prime p dividing νD, for any prime π dividing p, for any $r \in \nu \mathbf{Z}_p^*$, and for any $v_S \in W_{\nu D}$ that we have:*

$$\int_{W_p} \overline{S_{\Lambda,p}([w_p,0],r_p)}\,\tilde{\theta}(\beta[v_S+w_p,0])dw_p = \epsilon_p(k,\Lambda,\xi)\tilde{\theta}(\beta[w_S,0]).$$

Proof: Recall the definition of $\mathcal{S}_{\Lambda,\theta}$:

$$\mathcal{S}_{\Lambda,\theta}(a,w) = \Lambda(a_f)^{-1}\overline{S_{\Lambda,f}(w,\nu a_f \overline{a}_f/\mathcal{N}(a_f))}\tilde{\theta}(aw).$$

First we write the Siegel function appearing on the left side of the equation in Proposition 6.4 in terms of local Siegel functions. To this end, recall that for any prime p and any $r \in \mathbf{Z}_p^*$, we have $S_{\Lambda,p}([0,0],r) = 1-p^{-3k}$. Thus, if we let $r = \nu\zeta(\beta) \in \nu D \mathbf{Z}_f^*$ we can write for the left side of the equation in Proposition 6.4:
$\prod_{p \not| \nu D}(1-p^{-3k})^{-1} \int_{J_K \times W_{\nu D}} \mathcal{S}_{\Lambda,\theta}(\beta,w)dwd\beta =$

$$\prod_{p \not| \nu D}\left(\frac{\overline{S_{\Lambda,p}([w_p,0],r)}}{(1-p^{-3k})}\right) \int_{J_K \times W_{\nu D}} \prod_{p|\nu D} \overline{S_{\Lambda,p}([w_p,0],r_p)}\ \Lambda(\beta)^{-1}\ \tilde{\theta}(\beta[w,0])dwd\beta,$$

where we have used the fact that

$$\overline{S_{\Lambda,p}([w_p,t_p],\nu\zeta(\beta))}\,\tilde{\theta}(\beta[w_S+w_p,t_p])$$

is independent of $t_p \in \mathbf{Q}_p$. Thus, to prove Proposition 6.4 it will suffice to show that

$$\int_{J_K} \Lambda(\beta)^{-1} \int_{W_{\nu D}} \prod_{p|\nu D} \overline{S_{\Lambda,p}([w_p,0],\nu\zeta(\beta))}\ \tilde{\theta}(\beta[w,0]) =$$

$$\left(\prod_{p|\nu D} \epsilon_p(k,\Lambda,\xi)\right) \int_{J_K} \Lambda(\beta)^{-1}\tilde{\theta}(\beta)d\beta,$$

and this follows by induction from the formula stated in this lemma. **Q.E.D.**

6.4.2. The case of p inert, dividing ν

Since p is inert in K, Lemma 6.4.1 implies that we must prove that for any $v \in W_{\nu D}$, and $r \in \nu \mathbf{Z}_p^*$, we have

$$\int_{W_p} \overline{S_{\Lambda,p}([w_p,0],r)}\,\tilde{\theta}(\beta[v+w_p,0])dw_p = \epsilon_p(k,\Lambda,\xi)\tilde{\theta}(\beta[v,0]),$$

where ϵ_p is given by

$$\epsilon_p = (1+p^{-(3k-1)})\,(1-p^{-3k}) \qquad \text{(c.f. §6.1.4)}.$$

To evaluate this integral over W_p, we will decompose the integral into integrals over the regions $C_a = \pi^{-a}O_p^* + O_p$, for $a \geq 0$, and it will suffice to show that

$$\int_{C_a} \overline{S_{\Lambda,p}([w_p,0],r)}\,\tilde{\theta}(\beta[v+w_p,0])dw_p = \sigma_a\tilde{\theta}(\beta[v,0]),$$

where the σ_a are constants such that $\sum_{a\geq 0}\sigma_a = \epsilon_p(k,\Lambda,\xi)$. In fact, we will show that $\xi_a = 0$ for all $a \neq 0,1$ and will easily evaluate σ_0 and σ_1.

From our explicit evaluation of the local Siegel function (Proposition 5.4.5) we see that for all $w_p \in C_a$ we have

$$S_{\Lambda,p}([w_p, w_p\overline{w}_p D/2], r) = q^{2a}\beta_1(r/p^{2a}) + \sum_{j=2a+1}^{\infty} q^j\beta_1^*(r/p^j),$$

where $q = p^{-(3k-1)}$, where β_1 is the characteristic function of O_p, and where $\beta_1^*(x) = \beta_1(x) - p^{-1}\beta_1(px)$ (c.f. §5.3). Thus, if $a > ord_p(\nu)/2$, then $\beta_1(r/p^{2a}) = 0$ and $\beta_1^*(r/p^j) = 0$, so that $\sigma_a = 0$.

So we may assume that $a \leq ord_p(\nu)$. Since θ is primitive it satisfies the relation $l(a^{-1})\theta = 0$ for all finite integral ideles a. This implies that if $d = ord_p(\nu)$, then for every integer $b \geq 0$ and every $m \in W_f$, we have

$$\int_{p^{-b}O_p^*} \tilde{\theta}(m[w_p, w_p\overline{w}_p D/2])dw = \begin{cases} \tilde{\theta}(m) & \text{if } b = 0, \\ -\tilde{\theta}(m) & \text{if } b = 1, \\ 0 & \text{if } 2 \leq b \leq d/2. \end{cases}$$

So $\sigma_a = 0$ if $2 \leq a \leq d/2$ and

$$\sigma_0 = 1 + \sum_{j=1}^{\infty} q^j\beta_1^*(r/p^j), \qquad \text{and}$$

$$\sigma_1 = -q^2\beta_1(r/p^2) - \sum_{j=3}^{\infty} q^j \beta_1^*(r/p^j).$$

Combining the results of the two previous paragraphs, we see that $\sigma_a \neq 0$ only if $a = 0$ or 1, and so the integral in question is

$$\int_{W_p} \overline{S_{\Lambda,p}([w_p, w_p\overline{w}_p D/2], r)}\ \tilde{\theta}(\beta[w_S + w_p, w_p\overline{w}_p D/2])dw_p = \epsilon_p'\tilde{\theta}(\beta[w_S, 0]),$$

where

$$\begin{aligned}\epsilon_p' = \sigma_0 + \sigma_1 &= 1 + q\beta_1^*(r/p) + q^2(\beta_1^*(r/p^2) - \beta_1(r/p^2)) \\ &= (1 + p^{-(3k-1)})\ (1 - p^{-3k}) = \epsilon_p.\end{aligned}$$

Q.E.D.

6.4.3

The case where p splits in K and divides ν Let p be a prime that splits in K and which divides ν, and let π be a generator of the prime ideal of O_p, e.g. $\pi = pe + \overline{e}$. By Lemma 6.4.1 what we must prove in this case is that, for any $v \in W_{\nu D}$ and any $r \in \nu\mathbf{Z}_p^*$, we have

$$\int_{W_p} \overline{S_{\Lambda,p}([w_p, 0], r)}\,\tilde{\theta}(\beta[v + w_p, 0])dw_p = \epsilon_p(k, \Lambda, \xi)\tilde{\theta}(\beta[v, 0]),$$

where the constant $\epsilon_p(k, \Lambda, \xi)$ is given by (c.f. §6.1.4).

$$\epsilon_p(k; \Lambda, \xi) = \left(1 - p^{-(3k-1)}\right)\ \left(1 - \overline{\chi}_1^*(\pi)p^{-3k/2}\right)\ \left(1 - \overline{\chi}_1^*(\overline{\pi})p^{-3k/2}\right).$$

As in the case where p did not divide ν we will decompose the integral into a sum of integrals over the disjoint open sets $C_{a,b}$ for all $a, b \geq 0$, and as in the case of p inert, we use the primitivity of θ to simplify the integrals over the regions $C_{a,b}$. Thus, it will suffice to show that for some constants $\sigma_{a,b}$ we have

$$\int_{C_{a,b}} \overline{S_{\Lambda,p}([w_p, 0], r)}\,\tilde{\theta}(\beta[v + w_p, 0])dw_p = \sigma_{a,b}\tilde{\theta}(\beta[v, 0]),$$

and that $\sum_{a,b\geq 0} \sigma_{a,b} = \epsilon_p(k, \Lambda, \xi)$. The evaluation of these integrals over $C_{a,b}$ divides naturally into three cases, according as $b = 0$, or $a = 0$, or neither is zero. We will see that all of the terms $\sigma_{a,b}$ vanish unless $a, b \leq 1$, and so we will then only need to show that

$$\epsilon_p(k, \Lambda, \xi) = \sigma_{0,0} + \sigma_{0,1} + \sigma_{1,0} + \sigma_{1,1}.$$

We will evaluate these nonzero $\sigma_{a,b}$ terms using the explicit formula for the local Siegel function and the primitivity of θ.

Indeed, we state for future reference the primitivity condition satisfied by θ that we will use. Since θ is primitive, and $p|\nu$, we know (by Prop. 4.2a) that

$$l((\overline{\pi}/\pi)^{-a})\theta = l(\pi^a)l(\overline{\pi}^{-a})\theta = 0,$$

for all $a \neq 0$, and this implies, for any $m_f \in M_f$, that

$$\int_{C_{a,0}} \tilde{\theta}(m_f[w_p, -w_p\overline{w}_p\tau_1/2])dw_p = \begin{cases} \theta(m_f) & \text{if } a = 0, \\ -\theta(m_f) & \text{if } a = 1, \\ 0 & \text{if } a > 1 \ . \end{cases}$$

We will consider separately the integrals over $C_{a,0}$, $C_{0,b}$, and $C_{a,b}$ for $a, b > 0$.

We consider first the integral over $C_{a,0}$ for $a > 0$. The value of the local Siegel function on $C_{a,0}$ has been computed in Proposition 5.5.5, and states that if $w_p \in C_{a,0}$ then

$$S_{\Lambda,p}([w_p, -w_p\overline{w}_p\tau_1/2], r) = \epsilon'_{a,0}(\chi_1, r) + \epsilon_{a,b}(r),$$

where the constants ϵ' and ϵ are given by

$$\epsilon'_{a,0}(\chi_1, r) = \overline{\chi}_1(\pi)^a \left(1 + \sum_{j=1}^{a} (p\chi_1(\pi))^j \beta_1^*(r/p^j)\right) - q^a\beta_1(r/p^a),$$

and

$$\epsilon_{a,0}(r) = q^a\beta_1(r/p^a) + \sum_{c=a+1}^{\infty} q^c\beta_1^*(r/p^c).$$

Thus, we can remove the local Siegel function from integrand of the integral in question, and obtain

$$\int_{C_{a,0}} \overline{S_{\Lambda,p}([w_p, -w_p\overline{w}_p\tau_1/2], r)}\, \tilde{\theta}(\beta[w_S + w_p, -w_p\overline{w}_p\tau_1/2])dw_p =$$

$$\left(\overline{\epsilon'_{a,0}(\chi_1, r) + \epsilon_{a,0}(r)}\right) \int_{C_{a,0}} \tilde{\theta}(\beta[w_S + w_p, -w_p\overline{w}_p\tau_1/2])dw_p,$$

where $\chi_1^*(z) = \Lambda_Z^2\Lambda_T^{*3}(z)$ and β_1, β_1^* are the simple exponential sums defined in §5.3. The integral on the right side of this equation can be evaluated using the primitivity of the theta function. This allows us to evaluate $\sigma_{a,0}$ and we find

$$\sigma_{a,0} = \begin{cases} \overline{\epsilon_{0,0}(r)} & \text{if } a = 0 \ , \\ -\left(\overline{\epsilon'_{1,0}(\chi_1, r) + \epsilon_{1,0}(r)}\right) & \text{if } a = 1 \ , \\ 0 & \text{if } a > 1 \ . \end{cases}$$

A similar calculation holds for $\sigma_{0,b}$ and shows that $\sigma_{0,b} = \overline{\sigma_{b,0}}$.

Thus, we need only evaluate $\sigma_{a,b}$ in the case $a, b \geq 1$, and find the sum of the $\sigma_{a,b}$. So we assume that $a, b \geq 1$ from now on. In this case, our explicit evaluation of the local Siegel function (Prop. 5.5.5) yielded

$$S_{\Lambda,p}([w_p, -w_p\overline{w}_p\tau_1/2], r) = \epsilon'_{a,b}(\chi_1, r) \ + \ \epsilon_{a,b}(r) \ + \ \lambda_p(-rw\overline{w}\tau_1)\overline{\epsilon'_{b,a}(\chi_1, r)},$$

where

$$\epsilon_{a,b}(r) = q^{a+b}\beta_1(r/p^{a+b}) + \sum_{c=a+b+1}^{\infty} q^c\beta_1^*(r/p^c),$$

and

$$\epsilon'_{a,b}(\chi_1, r) = \overline{\chi}_1(\pi)^a q^b \left(\beta_1(r/p^b) + \sum_{j=1}^{a} (p\chi_1(\pi))^j \, \beta_1^*(r/p^{j+b}) \right) - q^{a+b}\beta_1(r/p^{a+b}),$$

and $\chi_1(\pi) = \Lambda_Z^2\Lambda_T^{*3}(\pi)p^{3k/2}$. Therefore, the integral in question can be written as

$$\int_{C_{a,b}} \overline{S_{\Lambda,p}([w_p, 0], r)}\, \tilde{\theta}(\beta[v + w_p, 0])dw_p =$$

$$\left(\overline{\epsilon'_{a,b}(\chi_1, r) + \epsilon_{a,b}(r)}\right) I_{a,b} + \epsilon'_{b,a}(\chi_1, r)I'_{a,b},$$

where

$$I_{a,b} = \int_{C_{a,b}} \tilde{\theta}(\beta[v + w_p, -w_p\overline{w}_p\tau_1/2])dw_p,$$

and

$$I'_{a,b} = \int_{C_{a,b}} \lambda_p(rw\overline{w}\tau_1)\tilde{\theta}(\beta[v + w_p, -w_p\overline{w}_p\tau_1/2])dw_p.$$

Using the observation that $[we, 0][w\overline{e}, 0] = [w, -w\overline{w}\tau_1/2]$, we find that

$$I_{a,b} = \int_{p^{-a}\mathbf{Z}_p e} \int_{p^{-b}\mathbf{Z}_p \overline{e}} \tilde{\theta}(\beta[v, 0][w_1, 0][w_2, 0])dw_2dw_1,$$

and since θ is primitive the integral over w_2 is zero unless $b = 1$, in which case

$$I_{a,1} = \left(\overline{\epsilon'_{a,b}(\chi_1, r) + \epsilon_{a,b}(r)}\right) \int_{p^{-a}\mathbf{Z}_p e} -\tilde{\theta}(\beta[v, 0][w_1, 0])dw_1,$$

and again applying the primitivity of θ we find that this integral is zero unless a is also 1 in which case it has the value

$$\left(\overline{\epsilon'_{a,b}(\chi_1, r) + \epsilon_{a,b}(r)}\right) \tilde{\theta}(\beta[v, 0]).$$

Thus

$$I_{a,b} = \begin{cases} \overline{\epsilon'_{1,1}(\chi_1, r) + \epsilon_{1,1}(r)}\tilde{\theta}(\beta[v, 0]) & \text{if } a = b = 1\ , \\ 0 & \text{if } a > 1 \text{ or } b > 1\ . \end{cases}$$

$I'_{a,b}$ can be evaluated in the same way and yields:

$$I'_{a,b} = \begin{cases} \epsilon'_{1,1}(\chi_1, r)\tilde{\theta}(\beta[v, 0]) & \text{if } a = b = 1\ , \\ 0 & \text{if } a > 1 \text{ or } b > 1\ . \end{cases}$$

Combining the above results we see that if $a, b \geq 1$, then

$$\int_{C_{a,b}} \overline{S_{\Lambda,p}([w_p,0],r)}\,\tilde{\theta}(\beta[v+w_p,0])dw_p = \xi_{a,b}\tilde{\theta}(\beta[v,0]),$$

where

$$\xi_{a,b} = \begin{cases} \overline{\epsilon'_{1,1}(\chi_1,r)+\epsilon_{1,1}(r)} + \epsilon'_{1,1}(\chi_1,r) & \text{if } a=b=1 \ , \\ 0 & \text{if } a>0 \text{ or } b>0 \ . \end{cases}$$

Thus, we can now calculate the sum of the $\sigma_{a,b}$:
$\sum_{a,b\geq 0}\sigma_{a,b} = \sigma_{0,0}+\sigma_{1,0}+\sigma_{0,1}+\sigma_{1,1} =$

$$\overline{\epsilon_{0,0}(r)} \;-\; \left(\overline{\epsilon'_{1,0}(\chi_1,r)} + \overline{\epsilon_{1,0}(r)} + \overline{\epsilon_{0,1}(r)} + \epsilon'_{1,0}(\chi_1,r)\right)$$

$$+\overline{\epsilon'_{1,1}(\chi_1,r)} + \overline{\epsilon_{1,1}(r)} + \epsilon'_{1,1}(\chi_1,r),$$

and this simplifies to

$$(1-\overline{\chi}_1(\pi))\,(1-\overline{\chi}_1(\overline{\pi}))\,(1-q) = \epsilon_p(k,\Lambda,\xi).$$

Thus, the proposition is proved in the case where p splits in K.

6.4.4
The case of p ramified Let p be a prime that ramifies in K and let π be the prime of K dividing p. By Lemma 6.4.1, what we must prove in this case is that

$$\int_{W_p} \overline{S_{\Lambda,p}([w_p,0],r)}\,\tilde{\theta}(\beta[v_S+w_p,0])dw_p = \epsilon_p(k,\Lambda,\xi),$$

where $\epsilon_p(k,\Lambda,\xi)$ is (c.f. §6.1.4)

$$\begin{cases} \left(1-\Lambda_T^*(\pi)p^{-3k/2}\right)\ \left(1+\Lambda_T^*(\pi)p^{3k/2-1}\right) & \text{if } \pi \text{ is in } \Sigma, \\ \left(1-\Lambda_T^*(\pi)p^{-3k/2}\right) & \text{if } \pi \text{ is not in } \Sigma \ . \end{cases}$$

As in the previous two cases we will write the integral over W_p as a sum of integrals over the compact open sets $C_a = \pi^{-a}O_p^* + O_p$ for $a \geq 0$, and we will show that

$$\int_{C_a} \overline{S_{\Lambda,p}([w_p,0],\nu\zeta(\beta))}\,\tilde{\theta}(\beta[w_S+w_p,0])dw_p = \sigma_a\tilde{\theta}(\beta[w_S,0]),$$

where σ_a are constants such that $\sum_{a\geq 0}\sigma_a = \epsilon_p(k,\Lambda,\xi)$. In fact, we will show that $\sigma_a = 0$ unless $a = 0, 1$.

The first step is to recall the formula for the value of the local Siegel function on C_a. By Prop. 5.4.6, we see that if $w \in C_a$, then $S_{\Lambda,p}([w,-w\overline{w}D/4],r) = \epsilon_a(r)$

$$\epsilon_a(r) = \begin{cases} 1+\sum_{j=1}^{\infty} q^j\beta_1^*(r/p^j) & \text{if } a=0 \ , \\ \Lambda_T^*(\pi)q^{a-1}p^{-3k/2}\beta_1(r/p^{a-1}) + \sum_{j=a}^{\infty} q^j\beta_1^*(r/p^j) & \text{if } a>0 \ , \end{cases}$$

and observe that this formula implies that $\epsilon_a(r) = 0$ if $a > ord_p(r)+1$. Thus, it will suffice to show that for every $a \leq ord_p(r) + 1$ one has

$$\overline{\epsilon_a(r)} \int_{C_a} \tilde{\theta}(\beta[w_S + w_p, \omega_{w_p}])dw_p = \sigma_a \tilde{\theta}(\beta[w_S, 0]),$$

and that $\sum_{a=0}^{ord_p(r)+1} \sigma_a = \epsilon_p(k, \Lambda, \xi)$.

Since $\tilde{\theta}$ is right $U(L)_f$ invariant, we see that $\sigma_0 = \epsilon_0(r)$. To evaluate σ_a for $a > 0$, we will need to use two facts about θ. First since θ is an eigenfunction of l_π

$$l_\pi \theta = \begin{cases} \theta & \text{if } p \in \Sigma\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

and since θ is also primitive

$$l_\pi l(\pi^{-a})\theta = 0 \quad \text{if } p^a | \nu.$$

These two properties of θ are equivalent to the following two properties.

$$\int_{\pi^{-1} O_p^*} \tilde{\theta}(m[w, -w\overline{w}D/4])dw = \begin{cases} -\tilde{\theta}(m) & \text{if } p \notin \Sigma\ , \\ (p-1)\tilde{\theta}(m) & \text{if } p \in \Sigma\ ; \end{cases}$$

and

$$\int_{\pi^{-a} O_p^*} \tilde{\theta}(m[w, -w\overline{w}D/4])dw = \begin{cases} -\tilde{\theta}(m) & \text{if } a = 1 < ord_p(r)\ , \\ 0 & \text{if } 1 < a \leq 1 + ord_p(r)\ . \end{cases}$$

From the second property (primitivity) we see that $\sigma_a = 0$ unless $a = 0$ or 1 and that $\sigma_1 = -\overline{\epsilon_1(r)}$ if $ord_p(r) \geq 1$. From the first property we see that

$$\xi_1 = \begin{cases} -\overline{\epsilon_1(r)} & \text{if } p \notin \Sigma\ , \\ (p-1)\overline{\epsilon_1(r)} & \text{if } p \in \Sigma\ , \end{cases}$$

and this implies that if $ord_p(r) \geq 1$, then we can not have $p \in \Sigma$.

Consider first the case where p does not divide ν. Since $\epsilon_a(r) = 0$ for $a > 1$ in this case, we see that

$$\sum_{a=0}^{\infty} \sigma_a = \sigma_0 + \sigma_1 = \begin{cases} \overline{\epsilon_0(r)} - \overline{\epsilon_1(r)} & \text{if } p \notin \Sigma\ , \\ \epsilon_0(r) + (p-1)\overline{\epsilon_1(r)} & \text{if } p \in \Sigma\ , \end{cases}$$

which simplifies to prove the Proposition in this case.

Consider now the case where p divides ν. Since θ is primitive, if θ is non-zero, then we must have $p \notin \Sigma$. Thus, we may assume $p \notin \Sigma$. Since $\epsilon_a(r)$ is zero for $a > ord_p(r) + 1$, and $\sigma_a = 0$ for $1 < a \leq ord_p(r) + 1$, we have

$$\sum_{a=0}^{\infty} \sigma_a = \sigma_0 + \sigma_1 = \overline{\epsilon_0(r)} - \overline{\epsilon_1(r)} = 1 - \Lambda_T^*(\pi)p^{-3k/2}.$$

This completes the proof of Lemma 6.4.4. and also that of Proposition 6.4. **Q.E.D.**

CHAPTER 7

EULER PRODUCTS III: FORMAL DIRICHLET SERIES

In this section, we calculate the Hecke eigenvalues of the Eisenstein series E_Λ associated to a Hecke character of weight k on the maximal torus of $G_{\mathbf{A}}$. We can then apply our main theorem and Shintani's eigenfunction theorem to determine a complete formula for the Fourier-Jacobi coefficients of E_Λ. An easy corollary of our formula is the arithmeticity of the Eisenstein series, which will be deduced from results of Damerell and Siegel on special values of L-series in the next chapter.

7.1. A formula for the Fourier-Jacobi coefficients of E_Λ

Recall that any modular form $F \in A_k(L, \chi)$ has a unique representation as a series of the form (§4.6)

$$F = \sum_{\xi \in \Xi} \sum_{\theta \in \mathcal{B}_\xi} Z_\theta \cdot \theta, \qquad Z_\theta \cdot \theta = \sum_a c_\theta(a) l(a) \theta,$$

where $\mathcal{B}_\xi$ is a basis of the space V_ξ of primitive Shintani eigenfunctions and Z_θ is a formal Dirichlet series of the form

$$Z_\theta = \sum_a c_\theta(a) x_a.$$

Recall that Z_θ is an element of the power series ring $\mathcal{P}$ in the infinite set of indeterminates $\{x_\pi\}$ indexed by the prime ideals of K, and that for any integral ideal a, x_a is an abbreviation for the monomial

$$x_a = \prod_{\pi | a} x_\pi^{ord_\pi(a)}.$$

Furthermore, if F is a simultaneous eigenfunction of the Hecke operators $\{T_\pi\}$ (§4.5) with eigenvalues $\lambda = \{\lambda_\pi\}$, then each of the formal Dirichlet series Z_θ for $\theta \in V_\xi$ has the form

$$Z_\theta = c_\theta(1) Z_{\xi,\lambda},$$

where $Z_{\xi,\lambda}$ is an explicit formal Dirichlet series which depends only on ξ, λ, and the character χ. Moreover, as shown by Shintani, this formal Dirichlet series admits an Euler product (Theorem 4.7). In the previous Chapter we calculated the constants $c_\theta(1)$ for the Eisenstein series E_Λ. In this Chapter, we find the eigenvalues of the Eisenstein series and hence are able to apply Shintani's Eigenfunction Theorem and produce a complete formula for the Fourier-Jacobi coefficients of the Eisenstein series.

Theorem. 7.1. *Let Λ be a Hecke character of weight k on the maximal torus of $D_{\mathbf{A}}$ and let E_Λ be the associated Eisenstein series. Then the Shintani form of the Fourier-Jacobi series for E_Λ is*

$$E_\Lambda = \sum_{\xi \in \Xi} L_\xi(k, \Lambda) \left(Z_\xi(k, \Lambda) \cdot \theta_\xi \right), \qquad Z_\xi(k, \Lambda) = \sum_a z_\xi(a) x_a,$$

where the theta function $\theta_\xi \in V_\xi$ and the constant $L_\xi(k, \Lambda, \xi)$ are as in Theorem 6.1, and the formal Dirichlet series Z_ξ admits an Euler product expansion

$$Z_\xi(k, \Lambda) = \prod_p Z_{\xi,p}(k, \Lambda),$$

where the local Euler factors are given as follows in the three cases depending on whether p remains inert, ramifies, or splits in K:

1) *if p remains inert in K, then $Z_p(x_p) = \sum_n z(p^n) x_p^n$ and*

$$Z_p(x) = \frac{R_p(x)}{(1 - p^{3k-4}x)\ (1 - p^{-3k}x)},$$

where $R_p(x) = 1$ if $p|\nu$ and $R_p(x) = 1 - p^{-3}x$ otherwise.

2) *if p ramifies in K, then $Z_p(x_\pi) = \sum_n z(\pi^n) x_\pi^n$ and*

$$Z_p(x) = \frac{R_p(x)}{(1 - \Lambda_T^*(\pi) p^{(3k/2)-2} x)\ (1 - \Lambda_T^*(\pi) p^{-3k/2} x)},$$

where $R_p(x) = 1$ if $\pi \in \Sigma$ and $R_p(x) = 1 - p^{-1}x$ if $\pi \notin \Sigma$.

3) *if p splits in K, then $Z_p(x_\pi, x_{\overline{\pi}}) = \sum_{m,n} z(\pi^m \overline{\pi}^n) x_\pi^m x_{\overline{\pi}}^n$ and*

$$Z_p(x, y) = \frac{R_p(x, y)}{Q_p(x) \overline{Q}_p(y)},$$

where the polynomial Q_p is

$$\left(1 - \frac{x}{\Lambda_T^*(\pi) p^{(3k/2)-2}}\right) \left(1 - \frac{x}{\overline{\Lambda_Z \Lambda_T^{*2}(\pi)} p^{-1}}\right) \left(1 - \frac{x}{\Lambda_T^*(\pi) p^{-3k/2}}\right),$$

and the polynomial $R_p(x, y)$ is $1 - p^{-2}xy$ if $p|\nu$. If p is prime to ν, then

$$R_p(x, y) = P_p(x) \overline{P}_p(y) + C_p xy,$$

the polynomial $P_p(x)$ is

$$P_p(x) = 1 - \left(\frac{\overline{\kappa}^*(\pi) \overline{z(\pi)}}{p^{1/2}} + \frac{\overline{\Lambda_Z}^2 \kappa^*(\pi)}{p^{3/2}} \right) x + \left(\frac{\overline{\kappa}^*(\pi)}{p^{5/2}} \right) x^2,$$

and the constant C is the real number given by

$$C = \frac{z(\pi)\Lambda_Z^2\overline{\kappa}^*(\pi)}{p^{3/2}} + \frac{\overline{z(\pi)}\,\overline{\Lambda_Z}^2\kappa^*(\pi)}{p^{3/2}} - \left(z(\pi)\overline{z(\pi)}p^{-1} + p^{-2}\right).$$

To complete the description of the numerator of Z_p, the coefficient $z(\pi)$ is characterized as which is the unique solution to the equation

$$\Lambda_Z(\overline{\pi})\kappa^*(\pi)p^{1/2}z(\pi) + \Lambda_Z(\overline{\pi})p^2\overline{z(\pi)} = \lambda_\pi - \Lambda_Z(\pi)\kappa^*(\overline{\pi})p^{1/2},$$

where $\lambda_\pi = \Lambda_Z^{-1}\Lambda_T^{*-1}(\pi)p^{3k/2} + \Lambda_Z\Lambda_T^{*2}(\pi)p + \Lambda_Z^{-1}\Lambda_T^{*-1}(\pi)p^{-3k/2+2}$.

This Theorem follows immediately from a calculation of the eigenvalues for the Eisenstein series (Lemma 7.3), combined with our main theorem (Theorem 6.1), and Shintani's eigenfunction theorem (Theorem 4.7). Thus, we need only show that E_Λ is a simultaneous eigenfunction of the Hecke operators and compute its Hecke eigenvalues.

7.2. The effect of the Hecke operators on Eisenstein series

In this section, we show that E_Λ is a simultaneous eigenfunction of the ring of Hecke operators defined in §4.5.

Lemma 7.2. *Notation as in §4.5. Let $\Lambda \in \mathcal{H}_k^*(D)$, then $E_\Lambda \in A_k(L, \Lambda_Z)$ and for all $\phi \in C_0(G_p, G(L)_p)$, $T_\phi E_\Lambda = \lambda_\phi E_\lambda$ where $\lambda_\phi = \int_{G_p} \Lambda\delta(x)\phi(x)dx$.*

Proof: Let C_p be the support of ϕ. Then

$$[T_\phi E_\Lambda](g) = \int_{C_p} \sum_{\gamma \in P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} \Lambda\delta(\gamma g x)\phi(x)dx.$$

Since C_p is a compact subset of G_p, we can interchange the integral and sum:

$$[T_\phi E_\Lambda](g) = \sum_{\gamma \in P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} T_\phi\Lambda(\gamma g),$$

where $T_\phi\Lambda(g) = \int_{C_p} \Lambda\delta(gx)\phi(x)dx$.

We will show that Λ is an eigenfunction of T_ϕ with eigenvalue λ_ϕ. Since ϕ is right $G(L)_p$ invariant and Λ is right $G(L)_f$ invariant, $T_\phi\Lambda$ is right $G(L)_f$ invariant. It is clear that $T_\phi\Lambda$ is in fact a continous function on $P_{\mathbf{Q}}\backslash G_{\mathbf{A}}/G(L)_f$ and that $T_\phi\Lambda$ has the same restriction to G_∞ as Λ, hence it is in $\mathcal{E}_k(G)$ (c.f. §1.8). By Lemma 1.8, $T_\phi\Lambda = \Lambda'\delta$, where $\Lambda' \in \mathcal{E}_k(D)$ is its restriction to $D_{\mathbf{A}}$. Thus, we should examine this restriction. Let $y \in D_{\mathbf{A}}$; then

$$T_\phi\Lambda(y) = \int_{C_p} \Lambda\delta(yx)\phi(x)dx = \Lambda(y)\int_{C_p} \Lambda\delta(x)\phi(x)dx = \lambda_\phi\Lambda(y).$$

Thus, $[T_\phi\Lambda] = \lambda_\phi\Lambda$, and therefore $T_\phi E_\Lambda = \lambda_\phi E_\Lambda$. **Q.E.D.**

7.3. The eigenvalues of Eisenstein series

In this section we use the previous Lemma and Shintani's explicit set of generators for the Hecke algebra to calculate the Hecke eigenvalues of the Eisenstein series.

Lemma 7.3. *Let p be a rational prime and π a prime of K dividing p. Let $\lambda_\pi = \lambda_{\phi_\pi}$, where ϕ_π is defined in §4.5, then*

$$\lambda_\pi = \begin{cases} p^{3k} + p + p^{-3k+4} & \textit{if } p \textit{ is inert in } K\ , \\ \Lambda_Z \Lambda_T^*(\pi) p^{3k/2} + \Lambda_Z(\pi) p + \Lambda_Z \Lambda_T^*(\pi) p^{-3k/2+2} & \textit{if } p \textit{ ramifies in } K\ , \\ \Lambda_Z^{-1} \Lambda_T^{*-1}(\pi) p^{3k/2} + \Lambda_Z \Lambda_T^{*2}(\pi) p + \Lambda_Z^{-1} \Lambda_T^{*-1}(\pi) p^{-3k/2+2} & \textit{if } p \textit{ splits in } K. \end{cases}$$

Proof: This is a straightforward calculation from the explicit formula for $S(\pi)$ given in the Corollary to Lemma 4.5, using the observation that if R_p is a compact subgroup of U_p containing $U(L)_p$, then

$$\int_{R_p d(z,x) G(L)_p} \Lambda\delta(g)dg = \Lambda_Z(z)\Lambda_T(x)[R_p d(z,x)U(L)_p : U(L)_p],$$

and so if π is a prime of K dividing a prime p of $\mathbf{Q}$, then

$$\lambda_\pi = \begin{cases} \Lambda_Z(\pi)\Lambda_T(\pi^{-1}) + \Lambda_Z(\pi)p + \Lambda_Z(\pi)\Lambda_T(\pi)p^4 & \text{if } p \text{ is inert in } K\ , \\ \Lambda_Z(\pi)\Lambda_T(\pi^{-1}) + \Lambda_Z(\pi)p + \Lambda_Z(\pi)\Lambda_T(\pi)p^2 & \text{if } p \text{ is ramified in } K\ , \text{ and} \\ \Lambda_Z(\overline{\pi})\Lambda_T(\pi^{-1}) + \Lambda_Z(\pi)\Lambda_T(\pi/\overline{\pi})p + \Lambda_Z(\overline{\pi})\Lambda_T(\overline{\pi})p^2 & \text{if } p \text{ is split in } K\ . \end{cases}$$

The lemma follows upon recalling that $\Lambda_T(\pi) = \Lambda_T^*(\pi)\mathcal{N}(\pi)^{3k/2}$ and observing (§1.6) that
$\Lambda_Z(\pi) = \Lambda_T^*(\pi) = 1$ if p is inert,
$\Lambda_Z(\pi), \Lambda_T^*(\pi) \in \{\pm 1\}$ if p is ramified, and
$\Lambda_Z(\overline{\pi}) = \Lambda_Z(\pi)^{-1}$ and $\Lambda_T^*(\overline{\pi}) = \Lambda_T^*(\pi)^{-1} = 1$ if $p = \pi\overline{\pi}$ is split.

Q.E.D.

CHAPTER 8

ARITHMETICITY

In this Chapter, we use the main theorem (Theorem 6.1) and our calculation of the Hecke eigenvalues of the Eisenstein series to show that the Fourier-Jacobi coefficients of Eisenstein series E_Λ are arithmetic theta functions the sense of Shimura [11].

8.1. Arithmetic theta functions

Arithmeticity for adelic modular forms is defined by placing an arithmeticity constraint on the values of their Fourier-Jacobi coefficients, which are adelic theta functions.

Shimura's definition of arithmeticity for classical theta functions is as follows [11]: Let $\mathcal{V}_\nu(\beta)$ be the space of classical theta functions defined in §3.1. An element $g \in \mathcal{V}_\nu(\beta)$ is said to be arithmetic ([11]) if for all $w \in K$ we have

$$e(-\nu D H_\beta(w,w)/4i) g(w) \in \overline{\mathbf{Q}}.$$

The $\overline{\mathbf{Q}}$-vector space of arithmetic, classical theta functions of weight ν with respect to the ideal associated to the idele β is denoted $\mathcal{V}_\nu(\beta)_{\overline{\mathbf{Q}}}$. Similarly, we say that an element $g \in \mathcal{V}_\nu$ is arithmetic if all of its restrictions $g_\beta(w) = g(w,\beta)$ are arithmetic theta functions in $\mathcal{V}_\nu(\beta)$. We denote the $\overline{\mathbf{Q}}$-subspace of arithmetic theta functions in $\mathcal{V}_\nu$ by $\mathcal{V}_{\nu,\overline{\mathbf{Q}}}$.

We say that an adelic theta function $\theta \in V_{k,\nu}(L)$ is arithmetic if $\tilde{\theta}(m_f)$ is an algebraic number for all $m_f \in M_f$, and denote the $\overline{\mathbf{Q}}$-vector space of arithmetic, adelic theta functions of weight (k,ν) by $V_{k,\nu}(L)_{\overline{\mathbf{Q}}}$.

Lemma 8.1. *Let $\Theta_\Lambda : V_{k,\nu}(L) \to \mathcal{V}_\nu$ be the isomorphism given in Lemma 3.1.3. Then $\theta \in V_{k,\nu}(L)_{\overline{\mathbf{Q}}}$ if and only if $\Theta_\Lambda(\theta) \in \mathcal{V}_{\nu,\overline{\mathbf{Q}}}$.*

Proof: By definition, $\theta \in V_{k,\nu}(L)$ is arithmetic if and only if for all $\beta \in K_f$, and $u_f \in U_f$ we have $\tilde{\theta}(u_f\beta) \in \overline{\mathbf{Q}}$. Since $U_f = (U_{\mathbf{Q}}U_\infty \cap U_f)U(\beta L)_f$, we may assume that u_f is the projection of some $u \in U_{\mathbf{Q}}$ into U_f, and hence that $\theta(u_f\beta) = \theta(u_\infty^{-1}\beta)$. Thus, θ is arithmetic if and only if for all $\beta \in K_f^*$ and all $w \in K^*$ we have

$$e(\nu\|\beta\||w_\infty|^2\sqrt{-D}/2)\, \tilde{\theta}([w_\infty,0]\beta) \in \overline{\mathbf{Q}};$$

and since $e(\nu\|\beta\||w_\infty|^2\sqrt{-D}/2) = e(-\nu D H_\beta(w,w)/4i)$ and

$$\Theta_\Lambda(\theta)(w,\beta) = \Lambda(\beta)^{-1}\tilde{\theta}([w_\infty,0]\beta),$$

we see that θ is arithmetic if and only if $\Theta_\Lambda(\theta) \in \mathcal{V}_{\nu,\overline{\mathbf{Q}}}$. **Q.E.D.**

8.2. Arithmeticity of Shintani's operators

Next, we observe that the operators $l(x)$ and l_π are defined over $\overline{\mathbf{Q}}$ and so we can construct a basis of arithmetic theta functions which are eigenfunctions of Shintani's representation. Indeed, this is obvious since the values of $[l(x)\theta]\tilde{}$ and $[l_\pi\theta]\tilde{}$ at a point $m_f \in M_f$ are finite linear combinations (with rational coefficients) of the values of $\tilde{\theta}$ at other points of M_f. We formulate this observation as the following Lemma:

Lemma 8.2.

a) *For any $x \in K_f$, $l(x) : V_{k,\nu}(L) \to V_{k,\nu\mathcal{N}(x)}(L)$ is defined over $\overline{\mathbf{Q}}$.*
b) *For any ramified prime π of K, l_π is defined over $\overline{\mathbf{Q}}$.*
c) *$V_{k,\nu}(L)_{\overline{\mathbf{Q}}}$ is an invariant $\overline{\mathbf{Q}}$-subspace of the representation l;*
d) *$V_{k,\nu}(L)_{\overline{\mathbf{Q}}}$ admits an orthogonal basis of eigenfunctions of the representation l.*

8.3. Arithmeticity of the Siegel inner product

Although the Siegel inner product of two arithmetic theta functions is not an algebraic number, it can be normalized to have this property. This normalization makes use of the value of Dedekind's eta function

$$\eta(z) = e(z/24)\prod_{n=1}^{\infty}(1 - e(nz)),$$

which is a modular form of weight 1/2 for a congruence subgroup of $SL_2(\mathbf{Z})$.

Lemma 8.3. *The Siegel norm $\|\theta\| = (\theta, \theta)$ of any arithmetic theta function $\theta \in V_{k,\nu}(L)_{\overline{\mathbf{Q}}}$ is in the set $\pi^{-3k}\eta(\sqrt{-D})^{-2}\overline{\mathbf{Q}}$, and hence the normalized Siegel inner product*

$$(\theta, \theta')_0 = \pi^{3k}\eta(\sqrt{-D})^2(\theta, \theta'),$$

is defined over $\overline{\mathbf{Q}}$.

Proof: Recall that there is an isomorphism $\Theta_\Lambda : V_{k,\nu}(L) \cong \mathcal{V}_\nu$ (Lemma 3.1.3) and that

$$(\theta, \theta') = c_0(\Theta_\Lambda(\theta), \Theta_\Lambda(\theta')),$$

where $c_0 \in \pi^{-3k}\overline{\mathbf{Q}}$ is given explicitly in Proposition 3.3.2. Thus, since $\eta(\sqrt{-D}) \in \mathbf{R}$ we must show that for all g in an arithmetic basis of V_ν we have

$$\|g\eta(\sqrt{-D})\| \in \overline{\mathbf{Q}}.$$

Recall (§3.1.2) that if $\{\beta_i\}$ is a complete set of representatives for the ideal classes of K, then

$$\mathcal{V}_\nu \cong \bigoplus_i \mathcal{V}_\nu{}'(\beta_i),$$

via the map $g \mapsto (g_{\beta_i})_i$, and by definition this induces a corresponding isomorphism on the $\overline{\mathbf{Q}}$-subspaces of arithmetic theta functions. Therefore, since the (Siegel) norm $\|g\|$ on $\mathcal{V}_\nu$ is simply the sum of the Siegel norms $\|g_{\beta_i}\|_{\beta_i}$ of the classical theta functions g_{β_i}, we need only show for every ideal β and for all g in an arithmetic basis of $\mathcal{V}_\nu{}'(\beta)$ that

$$\|g\eta(\sqrt{-D})\| \in \overline{\mathbf{Q}}.$$

To prove this for all arithmetic g in $\mathcal{V}_\nu{}'(\beta)$, it will suffice to prove it for an arithmetic basis of $\mathcal{V}_\nu(\beta)$, which we do next.

Let θ_j be the basis of the space $\mathcal{V}_\nu(\beta)$ which was constructed in Lemma 3.2.2 and recall that the Siegel norms of these basis elements are algebraic numbers (Lemma 3.3.1). Since they form a basis for $\mathcal{V}_{\nu,\beta}$, it will suffice to show that $\theta_j/\eta(\sqrt{-D})$ is an arithmetic theta function for all β and j. This is well-known [11], and we sketch a proof below.

To show that $\theta_j/\eta(\sqrt{-D})$ is an arithmetic theta function, we must show that for all $w \in K$

$$e\left(\mu\frac{(w/\omega_2)^2 - (w/\omega_2)\overline{(w/\omega_2)}}{4iIm(\omega_1/\omega_2)}\right)\theta(\mu w/\omega_2, \mu\omega_1/\omega_2; r_\Omega + j/\mu, s_\Omega)/\eta(\sqrt{-D}) \in \overline{\mathbf{Q}},$$

where $\Omega = (\omega_1, \omega_2)$ is a basis for the ideal (β) and r_Ω, s_Ω are rational numbers as in Lemma 3.2.1.

For $z \in \mathbf{C}$, $\alpha = (a, b) \in \mathbf{Q}^2$ and $\gamma \in GL_2(\mathbf{Q})$, let $w = az + b$, and let $\Omega = (\omega_1, \omega_2)$ where ${}^t\Omega = \gamma^t(z, 1)$. It is well-known, and not hard to show, that the function

$$f_{\alpha,\gamma}(z) = e\left(\mu\frac{\left((w/\omega_2)^2 - (w/\omega_2)\overline{(w/\omega_2)}\right)}{4iIm(\omega_1/\omega_2)}\right)\theta(\mu w/\omega_2, \mu\omega_1/\omega_2; r_\Omega + j/\mu, s_\Omega)/\eta(z),$$

is a modular function of z for some congruence subgroup of $SL_2(\mathbf{Z})$ and that its Fourier expansions at the cusps have algebraic coefficients. Thus, it is an arithmetic modular function and so takes algebraic values at points z in imaginary quadratic fields. In particular, the value at $z = \sqrt{-D}$ is algebraic, and so by appropriate choice of α and γ we see that $\theta_j/\eta(\sqrt{-D})$ is arithmetic. **Q.E.D.**

8.4. Arithmeticity of the L-series $\overline{L_\xi(k, \Lambda)}$

In this section we will evaluate $L(k; \Lambda, \kappa)$ (up to $\overline{\mathbf{Q}}$) using results of Damerell on the special values of Hecke L-series and of Siegel on special values of Dirichlet L-series.

Lemma 8.4. *Let $\xi = (\nu, \kappa^*, \Sigma) \in \Xi_\nu$ be as in §4.4, let $\Lambda \in \mathcal{H}_k(D)$ be a Hecke character on D, and let $L_\xi(k, \Lambda)$ be the L-series defined in Theorem 6.1, and let η be Dedekind's eta function. Then $L_\xi(k, \Lambda) \in \pi^{-3k}\eta(\tau)^{-2}\overline{\mathbf{Q}}$.*

Proof: Recall that

$$L_\xi(k, \Lambda) = \frac{\alpha(k, \Lambda, \Sigma)}{4i\nu^2\sqrt{-D}^{3k+1}}\frac{L_K((3k-1)/2, \overline{\chi_2^*})}{L_{\mathbf{Q}}(3k-1, \chi_K)\, L_K(3k/2, \overline{\chi_1^*})},$$

where $\chi_1^* = \Lambda_Z^2 \Lambda_T^{*3}$ and $\chi_2^* = \Lambda_Z^2 \Lambda_T^* \overline{\kappa^*}$. These Hecke characters have weights $3k$ and $3k-1$ respectively. The first is unramified, and the second has conductor dividing $4\nu D$. In the notation of §1.6, $\chi_1^* \in \mathcal{H}_{3k,1}^*(K)$, and $\chi_2^* \in \mathcal{H}_{3k-1,C_\nu}^*$. From Damerell [4] we see that

$$L_K((3k-1)/2, \chi_2^*) \in \left(\pi\eta(\tau)^2\right)^{3k-1} \overline{\mathbf{Q}},$$

and

$$L_K(3k/2, \chi_1^*) \in \left(\pi\eta(\tau)^2\right)^{3k} \overline{\mathbf{Q}},$$

where η is Dedekind's eta function, and from Siegel [14] we have

$$L_{\mathbf{Q}}(3k-1, \chi_K) \in \pi^{3k-1} \overline{\mathbf{Q}}.$$

Hence, $L(k; \Lambda, \kappa) \in \pi^{-3k} \eta(\tau)^{-2} \overline{\mathbf{Q}}$. **Q.E.D.**

8.5. Arithmeticity of the Eisenstein series

Shintani has shown that the subspaces $V_\xi = V_{k,\nu}(L)^0_{\kappa,S}$ are one dimensional in the case $K = \mathbf{Q}(i)$ by explicitly determining which of these subspaces are non-zero and by comparing dimensions. We have seen that, in any case, the projection of the Fourier-Jacobi coefficient $E_{\Lambda,\nu}$ into V_ξ has the form $L_\xi(k, \Lambda)\theta_\xi$ where θ_ξ is the element in V_ξ dual to the linear functional

$$l_\Lambda(\theta) = \int_{CL_K} \Lambda(\beta)^{-1} \tilde{\theta}(\beta) d\beta,$$

restricted to V_ξ.

Proposition 8.5. *$E_{\Lambda,\nu}$ is an arithmetic theta function for all $\Lambda \in \mathcal{H}_k(D)$ and $\nu \in \mathbf{Z}$.*

Proof: First, observe that in the case $\nu = 0$, we have $E_{\Lambda,0}(m_f)\tilde{} = \Lambda\delta(m_f)$ which is an algebraic number for all $m_f \in M_f$. So we may assume $\nu > 0$.

By Shintani's theorem (cf. Thm. 4.7) on dependence of the oldform components on the eigenvalues and on the primitive components, and by our calculation of the eigenvalues of E_Λ (which are algebraic numbers), it will suffice to show that the primitive components of the Fourier-Jacobi coefficients of E_Λ are arithmetic theta functions. So, by our main theorem (Theorem 6.1), it will suffice to show that for all $\xi = (\nu, \kappa^*, \Sigma) \in \Xi$ the adelic theta function $L(k, \Lambda, \xi)\theta_\xi$ is arithmetic (c.f. Theorem 6.1).

Observe that θ_ξ is the projection of an element θ_Λ into V_ξ, where θ_Λ is the element dual to the linear functional l_Λ on $V_{k,\nu}(L)$. Thus, it will suffice to show that $L(k, \Lambda, \xi)\theta_\Lambda$ is an arithmetic theta function.

Let $\mathcal{C}$ be any orthogonal basis of $V_{k,\nu}(L)$, then

$$\theta_\Lambda = \sum_{\theta \in \mathcal{C}} \overline{l_\Lambda(\theta)} \theta / \|\theta\|.$$

In particular, if $\mathcal{C}$ is a basis of arithmetic theta functions, then $L(k, \Lambda, \xi)\theta_\Lambda$ is arithmetic only if $L(k, \Lambda, \xi) l_\Lambda(\theta)/\|\theta\|$ is an algebraic number for all θ in $\mathcal{C}$. Thus, it will suffice to show that

$$L(k, \Lambda, \xi) \overline{l_\Lambda(\theta)} / \|\theta\| \in \overline{\mathbf{Q}},$$

for all θ in a (not neccessarily orthogonal) arithmetic basis, $\mathcal{C}$ of $V_{k,\nu}(L)$.

So let θ be an arithmetic theta function in $V_{k,\nu}(L)$. Since $l_\Lambda(\theta) \in \overline{\mathbf{Q}}$, we must show

$$\|\theta\| \in L(k;\Lambda,\kappa)\overline{\mathbf{Q}}.$$

Since $L(k;\Lambda,\kappa) \in \pi^{-3k}\eta(\tau)^{-2}\overline{\mathbf{Q}}$, (Lemma 8.4) we must show that

$$\|\theta\| \in \pi^{-3k}\eta(\tau)^{-2}\overline{\mathbf{Q}},$$

for all θ in an arithmetic basis of $V_{k,\nu}(L)$, which was proved in Lemma 8.3. **Q.E.D.**

APPENDIX A

PROOF OF LEMMA 4.2(B)

This appendix contains a detailed proof of a result of Shintani [Shintani, Prop.3, p.47]. The proof follows Shintani's proof closely, but provides more details and motivation at several crucial points.

The lemma in question states that the eigencharacters of Shintani's representation on $V_{k,\nu}(L)$ are Hecke characters and provides a conductor C_ν for them. The lemma will be proved in three steps. First, the problem will be reduced to one of showing that the trace of a certain representation is constant. Next, this trace will be expressed as the value of an adelic integral using a variant of Selberg's trace formula. Finally, the adelic integral will be evaluated using formulas for Gaussian sums and some results of Weil on the Fourier transforms of second degree characters.

For any integer C, let $K^*_{1,C}$ denote the subgroup of K^* consisting of all elements x such that $x \equiv 1 mod^* C$. Also, let C_ν be the positive integer defined by

$$C_\nu = \begin{cases} \nu D & \text{if } D \text{ is odd,} \\ 4\nu D & \text{if } D \text{ is even.} \end{cases}$$

Since $-D$ is the discriminant of an imaginary quadratic field, we know that in the first case D is congruent to 3 modulo 4, whereas in the second $D = 4D_1$ where D_1 is congruent to either 1 or 2 modulo 4.

A.1. Reduction to a trace calculation

Lemma A.1.1. *To prove lemma 4.2(b) it will suffice to show that for all $x \in K^*_{1,C_\nu}$ we have*

$$Tr(l_2(x)) = dim(V_{k,\nu}(L)),$$

where $l_2(x) = \eta_1(x)^{-1} l_1((x/\overline{x}))$ and

$$\eta_1(x) = (x_\infty/|x_\infty|)^{2k-1} \prod_{p\, inert} (-1)^{ord_p(x)}.$$

Moreover, it will suffice to show that the trace of l_2 restricted to K^*_{1,C_ν} is a constant.

Proof: It is clear that κ^* is a character of $I(\nu D)$. To prove Lemma 4.2(b), it will suffice to show that if $x \in K^*$ and if $x \equiv 1\ mod^* C_\nu$, then

$$\kappa^*((x)) = (x_\infty/|x_\infty|)^{2k-1}.$$

Observe first that the generator of a principal ideal is determined only up to a unit of O_K, thus the definition of the map κ^* only makes sense if the equation

$$\omega^{2k-1} \equiv 1 \ mod^* C_\nu,$$

has a unique solution in O_K^*. This condition is easily seen to hold in the cases we are considering.

Suppose that the conditions of the lemma hold and view l_2 as a representation of the abelian group K^*_{1,C_ν}. Since the characters of a group are linearly independent, we see that if the trace of l_2 is constant then the representation $x \mapsto l_1(x/\overline{x})$ restricted to K^*_{1,C_ν} must be scalar multiplication by $\eta_1(x)$, and the lemma follows. **Q.E.D.**

Recall that $V_{k,\nu}(L)$ admits an orthogonal direct sum decomposition into subspaces V_β indexed by the ideal classes of K, where V_β is the subspace consisting of those theta functions supported on the coset $M_{\mathbf{Q}}\beta M_\infty M(L)_f$. Moreover, if $z \in K^*$ then $l_1(z)$ preserves the subspace V_β and so we can make the following reduction:

Lemma A.1.2. *To prove Lemma 4.2(b) it will suffice to show that for each $\beta \in K_f^*$ the function $z \mapsto l_2(z)|V_\beta$ is constant for $z \equiv 1 \ mod^* C_\nu$.*

A.2. A Trace Formula

In this section we show that the constancy of the trace of l_2 restricted to K^*_{1,C_ν} can be expressed as the constancy of a relatively simple integral over $K_{\mathbf{A}}$. This will be performed in four steps:

1) First we show that V_β is isometric to a space W_β of theta functions on $U_{\mathbf{Q}}\backslash U_{\mathbf{A}}/U(L)_f$.
2) Then, we show that (under this isomorphism), $l_1(x)$ can be viewed as a translation followed by a projection and hence is a Hecke operator in the classical sense.
3) Next, we show that the projection operator can be expressed as a kernel operator and in the process we explicitly construct a reproducing kernel for W_β.
4) Finally, we use the reproducing kernel to obtain a trace formula and transform this formula to an explicit integral over $K_{\mathbf{A}}$.

The final reduction is stated as Lemma A.2.8.

A.2.1. A related representation

In this section, we will define a Hilbert space $W_{k,\nu}(\beta)$ of theta functions which is isometric to V_β but which is defined on the double coset space $U_{\mathbf{Q}}\backslash U_{\mathbf{A}}/U(\beta L)_f$. Moreover, we will transport the representation $l_1|K^*$ from V_β to an easily defined representation in $W_{k,\nu}(\beta)$.

Let W_r denote the complex vector space of all continuous functions $\phi : U_{\mathbf{Q}}\backslash U_{\mathbf{A}} \to \mathbf{C}$ which satisfy the following three properties:

1) $\phi(nu) = \lambda(r \cdot n)\phi(u)$ for all $n \in N_{\mathbf{A}}$
2) for all fixed $u_f \in U_f$, the function

$$w_\infty \mapsto \phi([w_\infty, 0]u_f)e(-r\|w_\infty\|\sqrt{-D}/2),$$

is holomorphic.

3) ϕ is right $U(\beta L)_f$–invariant for some $\beta \in K_f$.

For each $\beta \in K_f^*$, let $W_\nu(\beta)$ denote the subspace of $W_{\nu\|\beta\|}$ consisting of the right $U(\beta L)_f$-invariant elements. This vector space is a Hilbert space with the inner product defined by

$$(\phi_1, \phi_2) = \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \left(\overline{\phi_1}\phi_2\right)(u)du.$$

Observe that this integral converges because the integrand is a continous function and the region of integration is compact (homeomorphic to $U(\beta L)\backslash U_\infty \times U(\beta L)_f$).

Lemma. A.2.2. *For each $\theta \in V_{k,\nu}(L)$, let $\phi_{\beta,\theta}(u) = \theta(u\beta)$. Then the map $\theta \mapsto \phi_{\beta,\theta}$ defines a similitude from V_β to $W_\nu(\beta)$.*

Proof: A simple calculation will suffice to verify this lemma. **Q.E.D.**

Lemma. A.2.3. *Let $x \in K^*$ with $\|x\| = 1$, and let $\theta \in V_\beta$. Then*

$$\phi_{l_1(x_f)\theta}(u) = \frac{\mathcal{N}(num(x_f))^{1/2}}{\|\beta\|}\; x_\infty^k \int_{U(\beta L)_f} \phi_\theta(xuvx^{-1})dv.$$

Proof: This is a straightforward consequence of the definitions. Indeed,

$$\phi_{l_1(x_f)\theta}(u) = \left(l_1(x_f)\theta\right)(u\beta) = \mathcal{N}(num(x_f))^{1/2} \int_{U(L_f)} \theta(u\beta v x_f^{-1})dv.$$

A simple calculation shows that for any $m \in M_{\mathbf{A}}$ and for any $x_\infty \in \mathbf{C}$ such that $x\overline{x} = 1$ we have $\theta(mx_\infty) = x_\infty^k\theta(m)$. Applying this observation and the left $M_{\mathbf{Q}}$-invariance of θ to the integral in question we find

$$\int_{U(L)_f} \theta(u\beta v x_f^{-1})dv = \int_{U(\beta L)_f} \theta(xuvx^{-1}\beta x_\infty)\frac{dv}{\|\beta\|}$$

$$= x_\infty^k \int_{U(\beta L)_f} \phi_\theta(xuvx^{-1})\frac{dv}{\|\beta\|},$$

which concludes the proof. **Q.E.D.**

Corollary. A.2.3. *Let $x \in K^*$, and let $l_{x,\beta}$ be the linear transformation defined by*

$$l_{x,\beta}(\phi_{\beta,\theta}) = \phi_{\beta,l_1(x)\theta},$$

for all $\theta \in V_\beta$. Then $l_{x,\beta}\phi(u) = P_\beta(l'_x(\phi))$ where P_β is the projection operator of $W_{\nu\|\beta\|}$ into $W_\nu(\beta)$ given by

$$P_\beta(\phi)(u) = \int_{U(\beta L)_f} \phi(uv)\frac{dv}{\|\beta\|} = \int_{\beta O_f} \phi(u[w, Dw\overline{w}/2])\frac{dw}{\|\beta\|},$$

and l'_x is the conjugation operator defined on $W_{\nu\|\beta\|}$ by

$$l'_x(\phi)(u) = x_\infty^k \mathcal{N}(num(x))^{1/2}\ \phi(xux^{-1}),$$

which maps $W_\nu(\beta)$ isomorphically onto $W_{\nu/\|x_f\|}(\beta/x_f)$.

Proof: This follows immediately from the previous lemma. **Q.E.D.**

A.2.2. Bergman kernels and projection operators

In this section we construct a Bergman kernel which will represent the projection operator of W_r into $W_\nu(\beta)$. First we define a "trivial" theta function which will be used to construct the Bergman kernel.

Define $\psi : U_{\mathbf{Q}}\backslash U_{\mathbf{A}}/U(\beta L)_f \to \mathbf{C}$ by $\psi(u) = \psi_\infty(u_\infty)\psi_f(u_f)$ where

$$\psi_\infty([w,t]) = \lambda_\infty(rt)e(rw\overline{w}\sqrt{-D}/2)$$

$$\psi_f([w,t]) = \begin{cases} \lambda_f(r(t - w\overline{w}D/2)) & \text{if } w \in \beta O_f\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

and observe that ψ is supported on $U(\beta L)_f U_\infty N_{\mathbf{A}}$ and it satisfies the following two properties for all $u \in U_{\mathbf{A}}$, $n \in N_{\mathbf{A}}$:

$$\psi(nu) = \lambda(rn)\ \psi(u), \qquad \psi(u^{-1}) = \overline{\psi(u)}.$$

Next we define a Bergman kernel for V_β. For $v \in U_{\mathbf{A}}$, let $\phi_v : U_{\mathbf{Q}}\backslash U_{\mathbf{A}}/U(\beta L)_f \to \mathbf{C}$ be the map defined by

$$\phi_v(u) = \sum_{x\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \psi(v^{-1}xu) = \sum_{w\in K} \psi(v^{-1}[w,0]u).$$

This sum clearly converges uniformly and absolutely on compact subsets of $U_{\mathbf{A}}$.

Lemma. A.2.4. *Let $\theta \in W_{\nu\|\beta\|}$ and $v \in U_{\mathbf{A}}$. Then $\phi_v \in W_\nu(\beta)$ and*

$$(\phi_v, \theta) = c\ (P_\beta\theta)(v),$$

for some constant c independent of v and θ. Moreover, if $\{\theta_i\}$ is any orthonormal basis of $W_\nu(\beta)$, then for any $u, v \in U_{\mathbf{A}}$ we have

$$\phi_v(u) = \sum_i \overline{\theta_i(v)}\theta_i(u).$$

Proof: To prove that ϕ_v is in $W_\nu(\beta)$ we must verify that it is defined on the double coset space $U_{\mathbf{Q}}\backslash U_{\mathbf{A}}/U(\beta L)_f$ and that it satisfies the two properties stated in section A.2.1. The only part of this verification which is not immediately clear is that for $u_f \in U_f$ fixed, the following function of $w_\infty \in \mathbf{C}$

$$\phi_v([w_\infty, 0]u_f)e(-r\|w_\infty\|\sqrt{-D}/2),$$

is holomorphic. Expanding this using the definition of ϕ_v, it becomes

$$\sum_{x\in K} \psi(v^{-1}[x,0][w_\infty,0]u_f)\; e(-r\|w_\infty\|\sqrt{-D}/2).$$

By the definition of ψ_f, it is clear that the summand is non-zero only when x lies in some lattice independent of w_∞. Thus, it will suffice to show that the summand, for fixed x, is a holomorphic function of w_∞. This verification can be performed by a simple calculation.

Next we show that for any $v \in U_f$ and any $\theta \in W_{\nu\|\beta\|}$, we have

$$(\phi_v, \theta) = c\,(P_\beta\theta)(v).$$

First, expand the definition of the inner product of ϕ_v and θ:

$$(\phi_v,\theta) = \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \overline{\phi_v(u)}\theta(u)du = \int_{N_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \overline{\psi(v^{-1}u)}\theta(u)du = \int_{N_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \overline{\psi(u)}\theta(vu)du.$$

Observe that the integrand is $N_{\mathbf{A}}$–invariant, so

$$(\phi_v,\theta) = \int_{N_{\mathbf{A}}\backslash U_{\mathbf{A}}} \overline{\psi(u)}\theta(vu)du$$

$$= \int_{\mathbf{C}} \psi_\infty([w_\infty,0]) \int_{\beta O_f} \theta(v[w_\infty,0][w_f, w_f\overline{w_f}D/2])dw_f dw_\infty.$$

The inner integral is the projection operator P_β, up to a constant, and so

$$(\phi_v,\theta) = \int_{\mathbf{C}} e(rw\overline{w}\sqrt{-D}/2)\;(P_\beta\theta)(v[w,0])dw$$

$$= \int_{\mathbf{C}} e(rw\overline{w}\sqrt{-D})\;(\widetilde{P_\beta\theta})(v[w,0])dw,$$

where $(\widetilde{P_\beta\theta})(v[w,0])$ is a holomorphic function in $w \in \mathbf{C}$. By the residue theorem it is easy to show that this integral evaluates to the value of the integrand at $w = 0$, up to a constant $c \in \mathbf{R}$, independent of v and θ. Thus

$$(\phi_v,\theta) = c\;(P_\beta\theta)(v).$$

Now let $\{\theta_i\}$ be an orthonormal basis of $W_\nu(\beta)$. By what we have just proved, it is immediate that

$$\phi_v(u) = c\sum_i \overline{\theta_i(v)}\theta_i(u),$$

for any $v \in U_f$ and any $u \in U_{\mathbf{A}}$.

Now, using the fact that $\overline{\phi_v(u)} = \phi_u(v)$, we find

$$\phi_u(v) = \overline{\phi_v(u)} = c\sum_i \overline{\theta_i(u)}\theta_i(v),$$

for all $u \in U_{\mathbf{A}}$ and all $v \in U_f$. Since $\phi_u \in W_\nu(\beta)$, it is continuous on $U_{\mathbf{A}}$; and hence ϕ_u is uniquely determined by its values on U_f. Therefore, the displayed equation holds for all $v \in U_{\mathbf{A}}$. This proves that for all $v \in U_{\mathbf{A}}$ and for all $\theta \in W_\nu(\beta)$, we have

$$(\phi_v,\theta) = c\;(P_\beta\theta)(v).$$

Q.E.D.

A.2.4. A variant of Selberg's trace formula

Lemma A.2.5. *Let l be any diagonalizable unitary transformation of $W_\nu(\beta)$ and let $\phi'_v = l(\phi_v)$. Then*

$$Tr(l) = c\int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \phi'_v(v)dv,$$

for some constant c independent of l.

Proof: Let θ_i be an orthonormal basis of $W_\nu(\beta)$ consisting of eigenfunctions of the representation l, and let $\{\alpha_i\}$ be the eigenvalues, i.e., $l(\theta_i) = \alpha_i\theta_i$. Then, by Lemma A.2.4, we have

$$\phi'_v = l(\phi_v) = l\left(c\sum_i \overline{\theta_i(v)}\theta_i\right) = c\sum_i \alpha_i\overline{\theta_i(v)}\theta_i.$$

Thus,

$$\int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \phi'_v(v)dv = c\sum_i \alpha_i \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \|\theta_i(v)\|dv = c\sum_i \alpha_i(\theta_i,\theta_i) = cTr(l).$$

Q.E.D.

Lemma. A.2.6. *Let $x \in K_1^*$, then*

$$Tr(l_1(x)|V_\beta) = c\, x_\infty^k \mathcal{N}(num(x))^{1/2} \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \sum_{z\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \psi(u^{-1}zxux^{-1})du,$$

where c is a constant independent of x. Moreover, the summand is $N_{\mathbf{A}}$–invariant as a function of u.

Proof: A simple calculation. Indeed, by the previous lemmas, we have

$$Tr(l_1(x)|V_\beta) = Tr(l_{x,\beta}) = c' \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} (l_{x,\beta}\phi_u)(u)du,$$

for some constant c' independent of x. Thus,

$$Tr(l_{x,\beta}) = \frac{x_\infty^k\, \mathcal{N}(num(x_f))^{1/2}}{\|\beta_f\|}\, c' \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \int_{U(\beta L)_f} \phi_u(xuvx^{-1})dvdu,$$

and using the series definition of ϕ_u, we find that

$$Tr(l_{x,\beta}) = x_\infty^k\, \mathcal{N}(num(x_f))^{1/2}\, \frac{c'}{\|\beta\|} \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \int_{U(\beta L)_f} \sum_{z\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \psi(u^{-1}zxuvx^{-1})dvdu.$$

Finally, using the left $U(\beta L)_f$-invariance of ψ we find

$$= x_\infty^k\, \mathcal{N}(num(x_f))^{1/2}\, \frac{c'}{\|\beta\|} \int_{U(\beta L)_f} \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \sum_{z\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \psi((uv)^{-1}zx(uv)x^{-1})dudv$$

$$= x_\infty^k \mathcal{N}(num(x_f))^{1/2}\, c' \int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}} \sum_{z\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \psi(u^{-1}zxux^{-1})du.$$

Q.E.D.

Lemma. A.2.7. *Let $x \in K_1^*$, then*

$$Tr(l_{x,\beta}) = x_\infty^k \mathcal{N}(num(x_f))^{1/2}\, c \int_{K_{\mathbf{A}}} \psi([w, -w\overline{w}\zeta\sqrt{-D}/2])dw,$$

where c is a constant independent of x, and where $\zeta = \frac{1+x}{1-x}$.

Proof: In the previous lemma we found a certain expresssion for $Tr(l_{x,\beta})$. Using the fact that the summand of that expression is invariant, as a function of u, under translation by $N_{\mathbf{A}}$, we find

$$Tr(l_{x,\beta}) = c\, x_\infty^k \mathcal{N}(num(x_f))^{1/2} \int_{U_{\mathbf{Q}}N_{\mathbf{A}}\backslash U_{\mathbf{A}}} \sum_{z\in N_{\mathbf{Q}}\backslash U_{\mathbf{Q}}} \psi(u^{-1}zxux^{-1})du.$$

If we let $u = [w', 0]$ and $z = [v, 0]$, this becomes

$$Tr(l_{x,\beta}) = c\, x_\infty^k \mathcal{N}(num(x_f))^{1/2} \int_{K\backslash K_{\mathbf{A}}} \sum_{v\in K} \psi([-w', 0]\,[v, 0]\,[xw', 0])dw'.$$

A simple calculation shows that

$$[-w', 0]\,[v, 0]\,[xw', 0] =$$

$$[v + (x-1)w',\ (-(1+\overline{x})v\overline{w}' + (1+x)\overline{v}w' + (1+x)(1-\overline{x})w'\overline{w}')\,\tau/2],$$

where $\tau = \sqrt{-D}$. Making the substitution $w' = w/(x-1)$ and using the fact that $\overline{\zeta} = -\zeta$, this becomes

$$[-w', 0]\,[v, 0]\,[xw', 0] = [v + w,\ -\left((v+w)\overline{(v+w)} - v\overline{v}\right)\left(\frac{1+x}{1-x}\right)\,\tau/2],$$

and so

$$[-w', 0]\,[v, 0]\,[xw', 0] = [v + w,\ -\left((v+w)\overline{(v+w)}\right)\,\zeta\tau/2]\ \ [0, v\overline{v}\,\zeta\tau/2].$$

Since $v\overline{v}\zeta\tau/2 \in \mathbf{Q}$, and ψ is $N_{\mathbf{Q}}$–invariant, we find that the expression in question can be written as an integral over $K_{\mathbf{A}}$ as follows:

$$Tr(l_{x,\beta}) = c\, x_\infty^k \mathcal{N}(num(x_f))^{1/2} \int_{K\backslash K_{\mathbf{A}}} \sum_{v\in K} \psi\left([v + w,\ -\left((v+w)\overline{(v+w)}\right)\,\zeta\tau/2]\right) dw$$

$$= c\, x_\infty^k \mathcal{N}(num(x_f))^{1/2} \int_{K_{\mathbf{A}}} \psi([w, -w\overline{w}\zeta\tau/2])dw,$$

where c is some constant independent of x. **Q.E.D.**

Combining the results of this section and the last, we have succeeded in making the following reduction of Lemma 4.2(b).

Lemma A.2.8. *To prove lemma 4.2(b) it will suffice to show that for any $\beta \in K_f^*$ and any $\nu \in \mathbf{Z}$, there is a constant c such that for any $z \in K^*$ with $z \equiv 1 \; mod^* 8\nu D$, if we let*

$$\rho = \frac{z+\overline{z}}{z-\overline{z}} \frac{\nu \|\beta\| \sqrt{-D}}{2},$$

then we have

$$\int_{K_{\mathbf{A}}} \psi_\beta([w,0])\lambda(w\overline{w}\rho)dw = c \prod_{p \; inert} (-1)^{ord_p(z)} \left(\frac{z}{|z|}\right)^{-1} \mathcal{N}(num(z/\overline{z}))^{-1/2}.$$

Proof: A straightforward calculation. Indeed, by Lemma A.1.2, we must show that there is a constant c_1 such that for all $z \in K^*$ with $z \equiv 1 \; mod C_\nu$, the trace of $l_2(z)$ restricted to V_β is c_1. Using the definition of l_2 in terms of l_1, and the relationship of $l_1(x)$ to $l_{x,\beta}$, we find that we must show that

$$Tr\left(l_{z/\overline{z},\beta}\right) = c_1 \left(z/|z|\right)^{2k-1} \alpha(z),$$

where

$$\alpha(z) = \prod_{p \; inert} (-1)^{ord_p(z)}.$$

Finally, we use the formula for $l_{x,\beta}$ developed in Lemma A.2.7:

$$Tr(l_{z/\overline{z},\beta}) = z/\overline{z}^k \, \mathcal{N}(num(z/\overline{z}))^{1/2} \, c \int_{K_{\mathbf{A}}} \psi([w, -w\overline{w}\zeta\sqrt{-D}/2])dw,$$

and we obtain the stated result by a straightforward calculation. **Q.E.D.**

A.3. Weil's representation and Gaussian sums

In this section we evaluate the integral of lemma A.2.8. The first step is to use results of Weil on the Fourier transforms of second degree characters to obtain the following lemma.

Lemma A.3.1. *Let $\rho \in K^*$, and let $\Psi(w) = \psi([w,0])$, then*

$$\int_{K_{\mathbf{A}}} \Psi(w)\lambda(w\overline{w}\rho)dw = c \int_{K_{\mathbf{A}}} \widehat{\Psi}(w)\lambda(w\overline{w}(-1/\overline{\rho}))dw,$$

where $\widehat{\Psi}$ is the Fourier transform of Ψ, defined by

$$\widehat{\Psi}(w) = \int_{K_{\mathbf{A}}} \Psi(v)\lambda(v\overline{w} + \overline{v}w)dv,$$

and c is some constant independent of ρ.

Proof: First observe that, from the definition of Ψ, it follows that Ψ belongs to the space of Schwartz functions on $K_{\mathbf{A}}$ [18, p.158]. Also observe that, for any $\rho \in K_{\mathbf{A}}^*$, the function

$$F_\rho(w) = \lambda(w\overline{w}\rho),$$

is a second degree character on $K_{\mathbf{A}}$ [18, p.158], that is, for any fixed $y \in K_{\mathbf{A}}$, the following function is a character:

$$x \mapsto F_\rho(x+y)\, F_\rho(x)^{-1}\, F_\rho(y)^{-1}.$$

Having made these observations, we can then apply Theorem 2 of [18, p. 161] to conclude that

$$((\Psi F_\rho))\hat{}= c\widehat{\Psi} * F'_\rho, \qquad (**)$$

where $*$ denotes convolution of functions on $K_{\mathbf{A}}$, and

$$F'_\rho(w) = F_\rho(w\rho^{-1})^{-1}\gamma(F_\rho)\|\rho\|^{-1/2}.$$

The number $\gamma(F_\rho)$ is an algebraic number given explicitly by a Gaussian sum, and the number c is a constant, independent of ρ, chosen so that

$$\widehat{\widehat{F}}(w) = cF(-w),$$

for all Schwartz functions F.

If we further assume, as in the statement of the lemma, that $\rho \in K^*$, then $\|\rho\| = 1$ and by [18, Prop.5, p.179] $\gamma(F_\rho) = 1$, so

$$F'_\rho(w) = F_\rho(w\rho^{-1})^{-1} = \lambda(-w\overline{w}\rho/(\rho\overline{\rho})) = F_{\rho^*}(w),$$

where $\rho^* = -1/\overline{\rho}$. Thus, if we evaluate $(**)$ at zero we obtain the formula stated in the Lemma:

$$\int_{K_{\mathbf{A}}} \Psi(w)F_\rho(w)dw = (\widehat{\Psi F_\rho})(0) = \left(\widehat{\Psi} * F'_\rho\right)(0) = \int_{K_{\mathbf{A}}} \widehat{\Psi}(w)\; F_{\rho^*}(-w)dw.$$

Q.E.D.

Observe that for all $w \in K_{\mathbf{A}}$ we have

$$\Psi(w) = \prod_v \Psi_v(w_v),$$

where the product is over all valuations v of K, and where the local functions Ψ_v are defined in section A.2. From this observation, it is easy to see that the Fourier transform $\widehat{\Psi}$ admits a similar local decomposition:

$$\widehat{\Psi}(w) = \prod_v \widehat{\Psi}_v(w_v),$$

where

$$\widehat{\Psi}_v(w) = \int_{K_v} \Psi_v(w')\lambda_v(w\overline{w}' + \overline{w}w')dw'.$$

The following Lemma, when combined with Lemma A.2.8, completes the proof of Lemma 4.2(b).

Lemma A.3.2. *There is a constant c such that for all $z \in K^*$ with $z \equiv 1 \; mod^* C_\nu$, we have*

$$\int_{K_{\mathbf{A}}} \psi_\beta([w,0])\lambda(w\overline{w}\rho)dw = c \prod_{p\; inert} (-1)^{ord_p(z)} \left(\frac{z}{|z|}\right)^{-1} \mathcal{N}(num(z/\overline{z}))^{-1/2},$$

where ρ is as in Lemma A.2.8 and where c is a constant independent of z.

The proof of this Lemma splits naturally into two parts, according as D is even or odd. We consider here the case of D even. The proof for D odd is entirely similar, and is left to the reader. The first step in the proof to consider the special case when $z \in O_K$, we will then use this result to obtain the proof for general z.

Lemma A.3.3. *Assume that D is even. Let $z \in O_K$, $z \equiv 1 \; mod C_\nu$, and let $b \in \mathbf{Z}$ be the unique integer such that $z = bz_1$ for some $z_1 \in O_K$ satisfying $z_1 + \overline{z}_1 > 0$ and $(z_1, \overline{z}_1) = 1$. Let ρ be as in Lemma 2.8, and let $\rho^* = -1/\overline{\rho}$, so that*

$$\rho^* = \frac{z - \overline{z}}{z + \overline{z}} \frac{2}{\nu\|\beta_f\|\sqrt{-D}}.$$

For any valuation v of K, let

$$I_v = \int_{K_v} \widehat{\Psi}_v(w)\lambda_v(w\overline{w}(-1/\overline{\rho}))dw,$$

where Ψ_v is as above. Then

i) if $v = \infty$, then $I_\infty = (1/8)\; (z + \overline{z})/z$;

ii) if $v = p$ divides C_ν, then $I_p = \|\sqrt{-D}\|_p^{-1}$;

iii) if $v = p$ is splits as $p = \pi\overline{\pi}$ and is prime to C_ν, then

$$I_p = \|(z_1 + \overline{z}_1)/2\|_p;$$

iv) if $v = p$ is inert and prime to C_ν, then

$$I_p = \|(z_1 + \overline{z}_1)/2\|_p (-1)^{ord_p((z_1 + \overline{z}_1)/2)}.$$

Proof: By explicitly evaluating the Fourier Transforms $\widehat{\Psi}$, and using the congruence condition on z, the problem is reduced to evaluating certain well–known Gaussian sums.

Indeed, in the case $v = \infty$, we find that

$$I_\infty = \int_{\mathbf{C}} \int_{\mathbf{C}} e(au\overline{u} + u\overline{w} + \overline{u}w + bw\overline{w})du dw,$$

where $a = \nu\|\beta\|\sqrt{-D}/2$, $b = \rho^*$. Letting $u = x + iy$ and $w = s + it$, we find that

$$I_\infty = \left(\int_{\mathbf{R}} \int_{\mathbf{R}} e(ax^2 + 2xs + bs^2)dxds \right) \left(\int_{\mathbf{R}} \int_{\mathbf{R}} e(ay^2 - 2yt + bt^2)dydt \right).$$

Each of these double integrals is easily seen to equal $(2\sqrt{1-ab})^{-1}$ where the principal branch of the square root is taken for $Im(a) > 0$ and $b \in \mathbf{R}$). Thus, we find that

$$I_\infty = \frac{1}{4(1-ab)} = \frac{1}{4\left(1 - (z-\overline{z})/(z+\overline{z})\right)} = \frac{1}{4}\frac{z+\overline{z}}{z}.$$

In the case of $v = p$ a prime, we find that

$$\widehat{\Psi}_p(w) = \int_{\beta_p O_p} \lambda_p(-\nu\|\beta\|v\overline{v}D/2)\ \lambda_p(v\overline{w} + \overline{v}w)dv.$$

Since we are assuming that D is even, we see that if $v \in \beta_p O_p$, then $-\nu\|\beta\|v\overline{v}D/2 \in \mathbf{Z}_p$, and so

$$\widehat{\Psi}_p(w) = \int_{\beta_p O_p} \lambda_p(v\overline{w} + \overline{v}w)dv = \|\beta_p\|_p \int_{O_p} \lambda_p(Tr(\beta_p v\overline{w}))dv.$$

From this it easily follows that

$$\widehat{\Psi}_p(w) = \begin{cases} \|\beta_p\|_p & \text{if } w \in (\overline{\beta}_p\sqrt{-D})^{-1}O_p\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Applying this we find

$$I_p = \|\beta_p\|_p \int_{(\overline{\beta}\sqrt{-D})^{-1}O_p} \lambda_p(w\overline{w}\rho^*)dw.$$

Letting $v = \overline{\beta}_p\sqrt{-D}w$, we find that

$$I_p = \|\sqrt{-D}\|_p^{-1} \int_{O_p} \lambda_p(v\overline{v}\tilde{\rho})dv,$$

where $\tilde{\rho} = \rho^*/(\beta_p\overline{\beta}_p D) \in \mathbf{Q}_p$.

In the case $p|\nu D$, we find that $\tilde{\rho} \in \mathbf{Z}_p$ and so $I_p = \|\beta_p\|_p$.

If p is split, then there exist idempotents e, $\overline{e}$ in O_p such that $O_p = \mathbf{Z}_p e + \mathbf{Z}_p \overline{e}$. Thus, if we let $v = v_1 e + v_2 \overline{e}$, we find that

$$I_p = \int_{\mathbf{Z}_p} \int_{\mathbf{Z}_p} \lambda_p(v_1 v_2 \tilde{\rho})dv_1 dv_2,$$

and this is readily evaluated, to yield

$$I_p = \begin{cases} 1 & \text{if } \tilde{\rho} \in \mathbf{Z}_p, \\ \|\tilde{\rho}\|_p^{-1} & \text{otherwise.} \end{cases}$$

Using the congruence conditions satisfied by z, we can then deduce that

$$I_p = \|\sqrt{-D}\|_p^{-1} \, \|z_1 + \overline{z}_1\|_p.$$

If p is inert in K, then $O_p = \mathbf{Z}_p + \mathbf{Z}_p\sqrt{-D}$, so let $v = v_1 + v_2\sqrt{-D}$, and observe that

$$I_p = \|\sqrt{-D}\|_p^{-1} \int_{\mathbf{Z}_p}\int_{\mathbf{Z}_p} \lambda_p((v_1^2 + v_2^2 D)\tilde{\rho}) dv_1 dv_2$$

$$= \|\sqrt{-D}\|_p^{-1} \left(\int_{\mathbf{Z}_p} \lambda_p(v_1^2 \tilde{\rho}) dv_1 \right) \left(\int_{\mathbf{Z}_p} \lambda_p(v_2^2 D \tilde{\rho}) dv_2 \right).$$

These new integrals are Gaussian sums. Indeed, for any positive integer q and any residue class a modulo q, the Gauss sum $G(a,q)$ is defined by

$$G(a,q) = \sum_{x=1}^{q} e(ax^2/q).$$

Assuming $\tilde{\rho} \notin \mathbf{Z}_p$, let $j = -ord_p(\tilde{\rho})$, and let a be the residue class of $p^j \tilde{\rho}$ modulo p^j. Using this notation, we can express I_p as a product of Gauss sums:

$$I_p = G(a, p^j) G(aD, p^j)/p^{2j},$$

Using the formulas for Gauss sums, we can then show that

$$G(a,p^j)G(aD,p^j) = p^{-j}\left(\frac{-D}{p}\right)^j = p^{-j}(-1)^j.$$

Using the congruence relation satisfied by z we can then show that $j = ord_p((z_1 + z_1)/2)$, which completes the proof of the lemma. **Q.E.D.**

We now complete the proof of Lemma A.3.2 by extending the results of the previous Lemma from z in O_K to general z.

First observe that, by multiplying z by an appropriate element $a \in \mathcal{N}(O_K)$ which is congruent to 1 mod C_ν, we may assume that $z \in O_K$. Let $z = bz_1$, with b and z_1 as in the statement of Lemma A.3.2. Also, for any $z \in K^*$, let $\alpha(z)$ be the number defined by

$$\alpha(z) = \prod_{p \ inert} (-1)^{ord_p(z)}.$$

Then

$$\alpha(z)\,\left(\frac{z}{|z|}\right)^{-1}\mathcal{N}(num(z/\overline{z}))^{-1/2}=\alpha(b)\,\left(\frac{z_1}{|z_1|}\right)^{-1}\mathcal{N}(num(z_1/\overline{z}_1))^{-1/2}.$$

Since $num(z_1/\overline{z}_1)=(z_1)$, it will therefore suffice to show that

$$\int_{K_{\mathbf{A}}}\psi_\beta([w,0])\lambda(w\overline{w}\rho)dw=c\;\alpha(b)\;z_1^{-1}.$$

Next, applying Lemmas A.3.1 and A.3.2, we find that

$$\int_{K_{\mathbf{A}}}\psi_\beta([w,0])\lambda(w\overline{w}\rho)dw=c'\;\frac{z+\overline{z}}{z}\;\prod_p\|(z_1+\overline{z}_1)/2\|_p\;\alpha((z_1+\overline{z}_1)/2),$$

where c' is some constant independent of z. We can then use the product formula and the fact that $z_1+\overline{z}_1>0$ to dispose of the terms involving $\|(z_1+\overline{z}_1)/2\|_p$ and thereby obtain

$$\int_{K_{\mathbf{A}}}\psi_\beta([w,0])\lambda(w\overline{w}\rho)dw=c\;z_1^{-1}\alpha((z_1+\overline{z}_1)/2),$$

where $c=2c'$ is some constant independent of z. Finally, observe that the following relation holds for any odd integer a relatively prime to D:

$$\alpha(a)=\prod_{p\;inert}(-1)^{ord_p(a)}=\left(\frac{-D}{|a|}\right),$$

which implies that $\alpha(a)$ depends only on the residue class of a modulo D. Thus, if we let $z=bz_1$ with b and z_1 as in Lemma A.3.2, then

$$b(z_1+\overline{z}_1)/2=(z+\overline{z})/2\equiv 1\;mod\,D,$$

and so $\alpha((z_1+\overline{z}_1)/2)=\alpha(b)$. Thus, we have shown that

$$\int_{K_{\mathbf{A}}}\psi_\beta([w,0])\lambda(w\overline{w}\rho)dw=c\;z_1^{-1}\;\alpha(b),$$

which, as noted above, proves the Lemma. **Q.E.D.**

APPENDIX B

ANNOTATED INDEX TO NOTATION.

Chapter 1. The Arithmetic of $GU(2,1)$.

1.1. $\mathbf{Q}$-*structures on* $GU(2,1)$.

$R \in GL_3(K)$ the hermitian matrix defined in §1.1

G: the group of unitary similitudes of R

$G_{\mathbf{Q}} = \{g \in GL_3(K) : {}^t\overline{g}Rg = \mu(g)R\}$

1.2. Bruhat decompositions.

P: the parabolic stabilizing $(1,0,0)$

U: the unipotent radical of P

$D \cong Z \times T$: the maximal torus of P

Z: the center of G

T: a rank 1 torus in D

$d(z,x)$: an element of D

N: the center of U

W: the quotient of U by its center

$[w,t]$: an element of U

$M = TU = UT$

$a[w,t] = d(a,1)[w,t] \in M$

ι: a representative of the nontrivial element of the Weyl group

1.3. Action on a hermitian symmetric domain.

$G_\infty = \{g \in GL_3(\mathbf{C}) : {}^t\overline{g}Rg = \mu(g)R\}$

$\mathbf{P}^j_{\mathbf{C}}$ complex projective space

$D(R) = \{\xi \in \mathbf{P}^2_{\mathbf{C}} : {}^t\overline{\xi}R\xi > 0\}$

$jac : G_\infty \times \mathbf{P}^2_{\mathbf{C}} \to \mathbf{C}^*$ the jacobian determinant

$o = (\sqrt{-D}/2, 0, 1) = D(R)$

1.4. The class number.

$L = O_K^3 \in K^3$

$G(L)$: the stabilizer of L in G.

$\mu(L)$ the norm of a lattice L

$[L_1/L_2]$ the elementary divisors of L_1 with respect to L_2.

1.5. Local Iwasawa decompositions.

$\mathbf{K}_\infty$ the maximal compact subgroup of G_∞ which stabilizes o

$\mathbf{K}^0_\infty$ the kernel of jac and det in $\mathbf{K}^0_\infty$

$\delta : G_{\mathbf{A}}/\mathbf{K}^0_{\mathbf{A}} \to D_{\mathbf{A}}/D(L)_f Z_1$ the Iwasawa function

1.6. Hecke characters.

$\mathcal{H}_k^*$: the set of unitary Hecke characters of weight k

$\mathcal{H}_{k,C}^*$: the set of unitary Hecke characters of weight k and conductor C

$\epsilon_{F,p}(s,\chi)$ the local L-factor of an L-series

$L_F(s,\chi)$ a Hecke L-series for a character χ on the ideals of F

1.7. Hecke characters on the maximal torus.

$\mathcal{E}_k(D)$ modular forms on $D_{\mathbf{A}}$

$\mathcal{H}_k(D)$ the set of Hecke characters on D

Λ_Z, Λ_T: the Hecke characters on K associated to a Hecke character Λ on D

1.8. Lifting Hecke characters.

$\mathcal{E}_k(G)$ a space isomorphic to the Eisenstein series on $G_{\mathbf{A}}$

Chapter 2. Adelic Eisenstein Series.

2.1. Adelic modular forms.

$A_k(L)$ the space of modular forms of weight k of $G_{\mathbf{A}}$ for $G(L)_f$

$A_k(L,\chi)$ modular forms which transform by χ under central translations

2.2. Adelic Eisenstein series.

E_Φ a general Eisenstein series on $G_{\mathbf{A}}$

2.3. Restriction to $M_{\mathbf{A}}$.

E_Λ the Eisenstein series associated to a Hecke character Λ of $D_{\mathbf{A}}$

$M_{\mathbf{A}} = T_{\mathbf{A}}U_{\mathbf{A}}$ the space on which Fourier-Jacobi coefficients are defined

2.4. Fourier-Jacobi coefficients.

F_ν the νth Fourier-Jacobi coefficient of $F \in A_k(L,\chi)$

$\beta(m) = \|\delta(m_f)\|$, so $\beta(au) = \mathcal{N}(a_f)^{-1}$

λ a continous character on $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}/N(L)_f$ such that $\lambda(t_\infty) = e(t_\infty)$

2.5. Holomorphicity of the coefficients.

$\tilde{F}_\nu(m) = j_{k,\nu}(m)^{-1}F(m)$

$F_\nu(w_\infty, m_f) = \tilde{F}([w_\infty, 0]m_f)$

$\xi_1(m_\infty)$, $\xi_2(m_\infty)$ defined by $m_\infty \cdot o = (\xi_1(m_\infty), \xi_2(m_\infty), 1)$

2.6. Adelic theta functions.

$j_{k,\nu}(m)$ the factor of automorphy for adelic theta functions

$\tilde{\theta} = j_{k,\nu}^{-1}\theta$

2.7. The Siegel-Baily-Tsao-Karel Integral.

$E_{\Lambda,\nu}$: the ν-th Fourier-Jacobi coefficient of E_Λ.

$S_\Lambda(m,r)$: the Siegel function

2.8. Comparison with the tube domain case.

Chapter 3. Theta Functions.

3.1. Adelic theta functions.

3.1.1. The definition and basic properties of adelic theta functions.

$V_{k,\nu}(L)$ space of adelic theta functions of level ν, weight k for the lattice $L = O_k^3$.

3.1.2. Theta functions on $\mathbf{C} \times I_K$

$\mathcal{V}_\nu(\beta)$ classical theta functions of level ν wrt the ideal β in K.

3.1.3. Theta functions on $\mathbf{C} \times CL_K$

$\mathcal{V}_\nu{}'(\beta)$ the subspace of O_K^* invariant functions
$\mathcal{V}_\nu$ classical theta functions of level ν with complex multiplication by K
$\Theta_\Lambda : V_{k,\nu}(L) \to \mathcal{V}_\nu$

3.2. Classical theta functions.

3.2.1. Riemann forms, second degree characters, and classical theta functions.
H_A the minimal Riemann form for the lattice A
$V(H, \psi, A)$ classical theta functions on A with R.F. H and second degree character ψ
H_β the minimal Riemann form on the ideal β
ψ_β a second degree character on β wrt H_β

3.2.2. An explicit basis for the space of classical theta functions.
$\theta(w, z; r, s)$ Riemann's theta function
$\phi(w, z; r, s)$ a simple transform of Riemanns theta function

3.3. The Siegel inner product.

3.3.1. The Siegel inner product on classical theta functions.
$(g, g')_\beta$ the Siegel inner product on $\mathcal{V}_\nu(\beta)$

3.3.2. The Siegel inner product on adelic theta functions.
(θ, θ') the Siegel inner product on $V_{k,\nu}(L)$
dm, dt, dw, da normalized Haar measures on M,N,W,T.
$c_0 = (\theta, \theta')/(\Theta_\Lambda(\theta), \Theta_\Lambda(\theta'))$

Chapter 4. Shintani's Eigenfunction Theorem.

4.1. Shintani operators and primitive theta functions.
$l(x) : V_{k,\nu}(L) \to V_{k,\nu\mathcal{N}(x)}(L)$ the Shintani operator associated to an ideal x
$V_{k,\nu}(L)^0$ the primitive theta functions

4.2. Shintani's representation.
I_ν^1: the group of (norm 1) ideals of K relatively prime to ν
$num(x)$: the numerator of the ideal x
$l_1(x) : I_\nu^1 \to Aut_{\mathbf{C}}(V_{k,\nu}(L)^0)$ Shintani's representation
$V_{k,\nu}(L)_\kappa$ the eigenspaces of l_1
$V_{k,\nu}(L)_\kappa^0$ the primitive component the eigenspace
κ^* the Hecke character of weight $-(2k-1)$ associated to the eigencharacter κ
C_ν the conductor of κ^*

4.3. An action at the ramified places.
$U(L)_p^*$ a subgroup of index p of $U(\pi^{-1}L)_p$ containing $U(L)_p$
l_π the projection associated to the ramified prime π
$V_{k,\nu}(L)_{\kappa,\Sigma}$ an eigenspace of l, and $\{l_\pi\}$
$\xi = (\nu, \kappa^*, \Sigma)$ a Shintani "eigenvalue"
Ξ, Ξ_ν the set of all eigenvalues
$V_\xi = V_{k,\nu}(L)_{\kappa,\Sigma}$ the associated eigenspace

4.4. A system of coordinates for modular forms.
$\mathcal{B}_\xi$ a basis for V_ξ
$\mathcal{B}$ the union of the $\mathcal{B}_\xi$

4.5. Hecke operators on modular forms.
dg_p Haar measure on G_p such that $measure(G(L)_p) = 1$

$C_0(G_p, G(L)_p)$ continuous compactly supported $G(L)_p$ bi-invariant functions on G_p
$T_\phi \in End_{\mathbf{C}}(A_k(L,\chi))$ a general Hecke operator
$R_p(L)$ a ring of Hecke operators
$S(\pi)$ a minimal $G(L)_p$ double coset in G_p
T_π a generator of the ring of Hecke operators
$\epsilon(g)$ the elementary divisors of g

4.6. The action of Hecke operators on Fourier-Jacobi coefficients.
$\Delta : G_p/G(L)_p \to P_p/P(L)_p$ an Iwasawa map
Δ_Z, $\Delta_T = \delta$, Δ_U, Δ_M related Iwasawa maps
$C(M_p/M(L)_p)$ continous functions on $M_p/M(L)_p$
$T_\phi \in End_{\mathbf{C}}(C(M_p/M(L)_p)$ a Hecke operator on $C(M_p/M(L)_p$
$T_\pi \in End_{\mathbf{C}}(\bigoplus_\nu V_{k,\nu}(L))$
R_p a compact subset of U_p containing $U(L)_p$
$\alpha(x) = measure(x^{-1}R_p x \cap U(L)_p)$
$mod(x) = \mathcal{N}(x)^{-2}$ the modulus of $x \in T_p$ acting on U_p

4.7. Formal Dirichlet series.
Z_θ the formal Dirichlet series associated to F and θ
$c_\theta(a)$ its coefficients
x_π an indeterminate labelled by a prime ideal of K
$\mathcal{P}$ the ring of formal power series in $\{x_\pi\}$
x_a a monomial in the $\{x_\pi\}$ associated to an integral ideal a
λ_π an eigenvalue

4.8. Shintani's eigenfunction theorem.
$Z_{\xi,\lambda}$ the formal Dirichlet series for an eigenfunction with eighenvalues λ, at $\theta \in V_\xi$
$Z_{\xi,\lambda,p}$ the local Euler factor of $Z_{\xi,\lambda}$ at p
$R(x_\pi)$, $Q(x_\pi)$, $P(x_\pi)$, C polynomials related to the local L-factor

Chapter 5. An Euler Product Factorization of the Siegel Function.

5.1. Local Siegel functions.
$S_\Lambda(m,r)$ the Siegel function
$S_{\Lambda,p}(m,r)$ its p-component
$S_{\Lambda,f}(m,r)$ its finite adelic component
$S_{\Lambda,\infty}(m,r)$ its archimedean component

5.2. Evaluation of the archimedean factor.
$j_k(m,r)$ essentially the factor of automorphy for theta functions
c_1 a constant

5.3. Exponential sums.
$\beta_1(r)$ the characteristic function of $\mathbf{Z}_p$
$\beta_1^*(r) = \beta_1(r) + p^{-1}\beta_1(pr)$
$\beta_x(y)$ an exponential sum over $x\mathbf{Z}_p$
$\beta_x^*(y)$ an exponential sum over $x\mathbf{Z}_p^*$

5.4. Evaluation of the local Siegel function at inert and ramified primes.

5.4.1. Preliminaries.
ω_p a generator of O_p

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