THE UNIVERSITY OF CHICAGO

ON THE FOURIER-JACOBI COEFFICIENTS OF CERTAIN EISENSTEIN SERIES ON A UNITARY GROUP

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE, 1986

INTRODUCTION

The purpose of this paper is to develop an explicit formula for the Fourier-Jacobi coefficients of certain adelic Eisenstein series on the non-tube domain $GU(2,1)$ by extending a technique used by Siegel [15] , Baily [1], Tsao [17], and Karel [6] in the context of rational tube domains. This work also relies on results of Shintani [13] concerning the Fourier-Jacobi coefficients of those modular forms on $GU(2,1)$ which are simultaneous eigenforms of the Hecke operators.

Our main theorem provides a formula for the Fourier-Jacobi coefficients of certain Eisenstein series E_Λ of weight k associated to Hecke characters Λ of the imaginary quadratic field K which is used to provide a $\mathbf{Q}$-structure for $GU(2,1)$. The formula is deduced from an expansion of the Eisenstein series which is closely related to the Fourier-Jacobi expansion and which has the form

$$E_\Lambda = \sum_{\xi \in \Xi} L_\xi(k,\Lambda)\ (Z_\xi(k,\Lambda)\cdot \theta_\xi)\,,$$

where the complex number $L_\xi(k,\Lambda)$ is an explicit monomial of Hecke and Dirichlet L-series, $\{\theta_\xi : \xi \in \Xi\}$ is an infinite collection of primitive adelic theta functions, and $Z_\xi(k,\Lambda)$ is a formal Dirichlet series of operators on the graded ring of theta functions which can be expressed in terms of the eigenvalues of E_Λ by a result of Shintani [13]. A consequence of this explicit formula is the arithmeticity (in the sense of Shimura [11]) of the Eisenstein series.

In the remainder of this Introduction, we give the definition of the Eisenstein series that are considered and, after describing Shintani's theory of adelic theta functions, we state the main theorem precisely and sketch its proof.

Let K be an imaginary quadratic extension of $\mathbf{Q}$ with discriminant $-D$, let $R \in GL_3(K)$ be a nondegenerate, indefinite, hermitian matrix, and let $G_\mathbf{Q}$ be the $\mathbf{Q}$-algebraic group of similitudes of R; that is the group of all g in $GL_3(K)$ such that ${}^t\overline{g}Rg = \mu(g)R$ for some rational number $\mu(g)$. The $\mathbf{Q}$-isomorphism class of $G_\mathbf{Q}$ depends on D but, as will be seen in Chapter 1, it is independent of R. Without loss of generality (Lemma 1.1), we can take for R the following matrix :

$$R = \begin{pmatrix} 0 & 0 & \sqrt{-D} \\ 0 & -D & 0 \\ -\sqrt{-D} & 0 & 0 \end{pmatrix}.$$

The group G_∞ of real points of G acts biholomorphically on the hermitian symmetric domain $\mathcal{D}(R) = \{\xi \in \mathbf{P_C}^2 : {}^t\overline{\xi}R\xi > 0\}$ and the jacobian determinant of the action of $g = (g_{ij})$ at $\xi = (z, w, 1)$ is given by

$$jac(g, \xi) = (g_{31}z + g_{32}w + g_{33})^{-3}det(g).$$

Let P be the parabolic subgroup of G which stabilizes the boundary point ${}^t(1, 0, 0)$. Then P is the semidirect product of a maximal torus D of G and its unipotent radical U. Moreover, $D \cong Z \times T$ where Z is the center of G and both $Z_\mathbf{Q}$ and $T_\mathbf{Q}$ are isomorphic to K^*. Let N denote the center of U, then $N_\mathbf{Q} \cong \mathbf{Q}$ and $U_\mathbf{Q}/N_\mathbf{Q} \cong K$.

Let L be the lattice O_K^3 in K^3. It will be shown that L is a maximal lattice in the sense of Shimura [11], and that the adele group $G_\mathbf{A}$ admits an Iwasawa decomposition, $G_\mathbf{A} = P_\mathbf{A}G(L)_f\mathbf{K}_\infty^0$, where $\mathbf{K}_\infty^0$ is a subgroup of the maximal compact subgroup $\mathbf{K}_\infty$ of G_∞ which stabilizes the point $o = (\sqrt{-D}/2, 0, 1)$ and that $\mathbf{K}_\infty^0 \cap P_\infty$ is a finite subgroup Z_1 of the center of G_∞. Thus, there is a well-defined map (§1.5)

$$\delta : G_\mathbf{A}/G(L)_f\mathbf{K}_\infty^0 \to D_\mathbf{A}/D(L)_f Z_1, \qquad g \in \delta(g)U_\mathbf{A}G(L)_f\mathbf{K}_\infty^0,$$

which will be used to define the Eisenstein series of $G_\mathbf{A}$.

We are concerned here with modular forms on the adele group $G_\mathbf{A}$, which are continuous complex valued functions, F, on $G_\mathbf{Q}\backslash G_\mathbf{A}/G(L)_f$ such that for each fixed $g_f \in G_f$, and an integer $k \geq 0$ called the weight of F, the function $jac(g_\infty, o)^{-k}F(g_\infty g_f)$ depends only on the point $g_\infty o \in \mathcal{D}(R)$ and induces a holomorphic function there. This space of modular forms of weight k is denoted $A_k(L)$. The center of $G_\mathbf{A}$ acts on $A_k(L)$ by translation and the eigencharacters of this representation are ideal class characters. The eigenspace of those modular forms which transform by χ is denoted $A_k(L, \chi)$.

Of particular interest to us are the Eisenstein series, which are modular forms of integer weight, say $k > 0$, associated to Hecke characters Λ of the same weight, k, on the maximal torus of G. These Eisenstein series are defined by the series

$$E_\Lambda(g) = \sum_{\gamma \in P_\mathbf{Q}\backslash G_\mathbf{Q}} \Lambda\delta(\gamma g).$$

The Hecke character Λ is a continuous homomorphism from the group $D_\mathbf{Q}\backslash D_\mathbf{A}/D(L)_f$ to $\mathbf{C}^*$ such that

$$\Lambda(d_\infty d_f) = jac(d_\infty, o)^k \Lambda(d_f).$$

The restrictions of Λ to Z and T are Hecke characters of K, denoted Λ_Z and Λ_T. Λ_Z is a character of the class group, and $\Lambda_T^*(x) = \Lambda_T(x)\|x\|^{3k/2}$ is a unitary Hecke character. The Eisenstein series are simultaneous eigenfunctions of the ring of Hecke operators, and their eigenvalues are computed in Chapter 7. The existence of local Iwasawa decompositions at all places of G implies that E_Λ is determined by its values on $M_\mathbf{A} = T_\mathbf{A}U_\mathbf{A}$.

If F is any modular form in $A_k(L, \chi)$, then its restriction to $M_\mathbf{A}$ admits a Fourier-Jacobi expansion of the form

$$F = \sum_{r \in \mathbf{Q}} F'_r,$$

where the functions F'_r are defined on $M_{\mathbf{A}}$ by

$$F'_r(m) = \int_{N_{\mathbf{A}}} F(nm)\lambda(-r\cdot n)dn.$$

The relationship between these Fourier-Jacobi coefficients and classical theta functions is rather complex. To describe this relationship, we must first partition $M_{\mathbf{A}}$ into countably many open subsets $M_{\mathbf{A}}(c)$,

$$M_{\mathbf{A}}(c) = \{m = ua : u \in U_{\mathbf{A}}, a \in T_{\mathbf{A}}, (a_f\overline{a}_f) = (c)\},$$

indexed by rational numbers c which are norms of ideals of K. Now, let $F'_{r,c}$ denote the restriction of F'_r to $M_{\mathbf{A}}(c)$. In Chapter 3 we will see that $F'_{r,c}$ is non-zero only if rc is a nonnegative integer and that $F'_{r,c}$ can be identified with a tuple of classical theta functions all of which have the same weight which is proportional to rc. Thus, for each rational number r, the Fourier-Jacobi coefficient F'_r is associated to an infinite collection of vectors of classical theta functions of varying weights.

Rather than study these somewhat complicated Fourier-Jacobi coefficients directly (which was Shintani's approach), we investigate a related expansion of F into a series of functions whose relationship with classical theta functions is more direct and simple. For each nonnegative integer ν, let F_ν be the complex valued function on $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}/M(L)_f$ defined by

$$F_\nu(ua) = \int_{N_{\mathbf{A}}} F(nua)\lambda(-\nu\|a_f\|\cdot n)dn.$$

Observe that if $\nu = rc$ then

$$F_\nu|M_{\mathbf{A}}(c) = F'_r|M_{\mathbf{A}}(c).$$

This implies that we obtain an expansion for F in terms of the F_ν:

$$F = \sum_{\nu=0}^{\infty} F_\nu.$$

This series converges uniformly and absolutely on compact subsets of $M_{\mathbf{A}}$. (Indeed, when both sides are restricted to $M_{\mathbf{A}}(c)$ for any fixed c, we obtain the restriction of the Fourier-Jacobi expansion to $M_{\mathbf{A}}(c)$, which converges normally.) To discriminate between this expansion and the Fourier-Jacobi expansion, we will call this expansion the *adelic theta expansion* of F and we call F_ν the νth *adelic theta coefficient*.

The coefficient F_ν belongs to a certain space $V_{k,\nu}(L)$ of adelic theta functions which is isomorphic to a finite direct sum of spaces of classical theta functions of weight proportional to ν. Thus, the functions F_ν can be identified with a vector of theta functions *without having to restrict* to $M_{\mathbf{A}}(c)$.

The space $V_{k,\nu}(L)$ of adelic theta functions consists of all continuous functions θ on the double coset space $M_{\mathbf{Q}}\backslash M_{\mathbf{A}}/M(L)_f$ which satisfy two conditions. First, for any n in the center $N_{\mathbf{A}}$ of the unipotent radical $U_{\mathbf{A}}$ of $M_{\mathbf{A}}$, one has

$$\theta(nua) = \lambda(\nu\|a_f\|\cdot n)\theta(ua),$$

where λ is the unique continuous character on $N_{\mathbf{Q}} \backslash N_{\mathbf{A}} / N(L)_f$ such that under the identification of N_∞ with $\mathbf{R}$, we have $\lambda(n_\infty) = e^{2\pi i n_\infty}$. To describe the second condition we must introduce some additional notation. Let ξ_1, ξ_2 be the complex valued functions defined on M_∞ by

$$m_\infty \cdot o = (\xi_1(m_\infty), \xi_2(m_\infty), 1),$$

where $o = (\sqrt{-D}/2, 0, 1)$ and let $j_k(m_\infty, r) = jac(m_\infty, o)^k \, e(r\xi_1(m_\infty))$. The second condition asserts that for $m = ua \in M_{\mathbf{A}}$ with m_f fixed, the function

$$\tilde{\theta}(m) = j_k(m_\infty, \nu \|a_f\|)^{-1} \, \theta(m),$$

depends only on the point $\xi_2(m_\infty) \in N_\infty \backslash M_\infty / T_\infty \cong \mathbf{C}$ and the function it defines on $\mathbf{C}$ is holomorphic.

In Chapter 3 we show that the space of adelic theta functions is isomorphic to the direct sum of finitely many spaces of classical theta functions, one for each ideal class in K. This isomorphism is made explicit and can be used to construct a basis for the adelic theta functions using Riemann's theta function. There is a natural inner product on this space of theta functions, called the Siegel inner product, which is defined by

$$(\theta, \theta') = \int_{M_{\mathbf{Q}} \backslash M_{\mathbf{A}}} \overline{\theta(m)} \theta'(m) dm,$$

for some right-invariant Haar measure dm on $M_{\mathbf{A}}$.

Shintani has studied a natural action of the idele group $K^*_{\mathbf{A}}$ on the spaces of adelic theta functions and has developed a theory of "newforms" for the graded ring $\bigoplus_\nu V_{k,\nu}(L)$ with respect to the Siegel inner product, which is similar to theory of newforms in the modular form case.

More precisely, Shintani defines a map, l, from the group I_K of ideals of K to the group of endomorphisms of the graded ring of theta functions. This map has the property that $l(a)$ maps theta functions of weight ν to theta functions of weight $\nu \mathcal{N}(a)$ where $\mathcal{N}(a)$ is the norm of the ideal a. The map l is defined on $T_f / T(L)_f \cong K^*_f / O^*_f \cong I_K$ by

$$\left(l(a)\theta\right)(m) = \int_{U(L)_f} \theta(mua^{-1}) du.$$

Observe that the operators $l(a)$ are injections for all integral ideals a. (Indeed, $\theta(mua^{-1}) = \theta(ma)$ for such a). The subspace of imprimitive theta functions is defined to be the space spanned by the images of the maps $l(a)$ where a ranges over all integral ideals not equal to O_K. The orthogonal complement of the space of imprimitive theta functions with respect to the Siegel inner product is called the space of primitive theta functions, and is denoted $V_{k,\nu}(L)^0$. One of Shintani's main theorems asserts that there is an orthogonal decomposition of the space of adelic theta functions into images of the primitive spaces under the maps $l(a)$. More precisely, it is shown that for every positive integer ν, there is a decomposition

$$V_{k,\nu}(L) = \bigoplus_{\substack{a \in I^+ \\ \mathcal{N}(a) | \nu}} l(a) \left(V_{k,\nu/\mathcal{N}(a)}(L)^0\right),$$

where the sum is orthogonal direct and I^+ denotes the monoid of integral ideals of K. Thus, a ranges over the finite set of all integral ideals of K whose norm divides ν. Thus, if $\mathcal{B}_\nu$ is a basis for the space of primitive theta functions of weight ν, and if $\mathcal{B}$ is the union of the $\mathcal{B}_\nu$, then a basis for the ring of adelic theta functions is given by $\{l(a)\theta : a \in I^+, \theta \in \mathcal{B}\}$.

Shintani also shows that the map $a \mapsto \mathcal{N}(a)^{1/2} l(a/\overline{a})$ defines a representation of the group of ideals prime to νD on the space of primitive theta functions of level ν, and hence induces an orthogonal decomposition of the space of primitive theta functions into the eigenspaces, $V_{k,\nu}(L)^0_\kappa$ of this representation, where the eigencharacters κ are essentially Hecke characters. More precisely, if κ is one of the eigencharacters, then

$$\kappa^*(a) = \kappa(a/\overline{a}) \prod_{p\ inert} (-1)^{ord_p(a)}$$

defines a homomorphism on the ideals of K relatively prime to νD which is induced from a Hecke character of conductor dividing $4\nu D$.

These eigenspaces can be further decomposed using a commuting set of projections, l_π, associated to the ramified primes π of K. These projections are diagonalizable endomorphisms with eigenvalues 0 and 1, and are defined by

$$\left(l_\pi \theta\right)(m) = \int_{U(L)_p^*} \theta(mu) du,$$

where p is the prime of $\mathbf{Q}$ divisible by π and $U(L)_p^* \supset U(L)_p$ is the subgroup of index p in $U(\pi^{-1}L)_p$ whose intersection with N_p is $N(L)_p$. The simultaneous eigenspaces associated to this set of projections can be classified by subsets Σ of the set of ramified primes for which the eigenvalues are 1. By intersecting these two decompositions, Shintani forms an orthogonal decomposition of the space of primitive theta functions of level ν into eigenspaces $V_{k,\nu}(L)^0_{\kappa,\Sigma}$. (He also shows that when K is the Gaussian number field these eigenspaces have dimension 1 over $\mathbf{C}$, but it is not known whether these eigenspaces have dimension 1 for other fields.)

Shintani's decomposition of the space of primitive theta functions can be used to define a system of coordinates for modular forms which has interesting arithmetic properties. To construct this system of coordinates, Shintani chooses a basis $\mathcal{B}$ of the space of primitive theta functions of all levels, subject to the restriction that all basis elements be in the eigenspaces $V_{k,\nu}(L)^0_{\kappa,\Sigma}$. More precisely, let Ξ_ν be the set of all triples $\xi = (\nu, \kappa^*, \Sigma)$ such that the eigenspace $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$ is nonzero, and let Ξ be the union of the Ξ_ν for all non-negative integers ν. Further, for any $\xi \in \Xi$, let $\mathcal{B}_\xi$ be an orthonormal basis of V_ξ (with respect to the Siegel inner product), and let $\mathcal{B}_\nu$ be the basis of the finite dimensional vector space $V_{k,\nu}(L)^0$ of primitive theta functions of weight ν defined by

$$\mathcal{B}_\nu = \bigcup_{\xi \in \Xi_\nu} \mathcal{B}_\xi.$$

Finally, let $\mathcal{B} = \bigcup_{\nu=0}^\infty \mathcal{B}_\nu$ as before. Once this choice of bases has been made, the adelic theta coefficients F_ν in the expansion

$$F = \sum_{\nu=0}^\infty F_\nu$$

of any modular form F can be written uniquely as a finite linear combination involving translates of elements of $\mathcal{B}$ by Shintani operators; that is,

$$F_\nu = \sum_{\substack{a \in I+ \\ \mathcal{N}(a)|\nu}} \sum_{\theta \in \mathcal{B}_{\nu/\mathcal{N}(a)}} z_\theta(a)\,(l(a)\theta)\,,$$

for some complex numbers $z_\theta(a)$ where a ranges over the monoid of integral ideals of K.

This expression can be considerably simplified by viewing the adelic theta expansion to be an element of the topological algebra $\hat{\mathcal{R}}$ whose underlying vector space structure is defined by

$$\hat{\mathcal{R}} = \prod_{\nu=0}^{\infty} V_{k,\nu}(L)$$

with the usual (Tychonoff) product topology, with componentwise addition, and whose multilication is given as follows:

Let $x = (x_0, x_1, \ldots), y = (y_0, y_1, \ldots) \in \hat{\mathcal{R}}$, then $x \cdot y = (z_0, z_1, \ldots)$, where

$$z_n = \sum_{i+j=n} x_i y_j$$

If we associate to each modular form F the element $(F_1, F_2, \ldots)$ in $\hat{\mathcal{R}}$, then we obtain a continuous homomorphism of the graded ring of modular forms into the ring $\hat{\mathcal{R}}$. In this light, the adelic theta expansion for F described above can be written more simply as

$$F = \sum_{\theta \in \mathcal{B}} \sum_{a \in I+} z_\theta(a)\,(l(a)\theta)\,,$$

where for each $\theta \in \mathcal{B}$ the inner sum clearly converges to an element of $\hat{\mathcal{R}}$ and the outer sum over $\mathcal{B}$ also converges in $\hat{\mathcal{R}}$, since only finitely many summands lie in any particular component $V_{k,\nu}(L)$. This expansion can be written more suggestively as

$$F = \sum_{\theta \in \mathcal{B}} (Z_\theta \cdot \theta)\,, \qquad Z_\theta \cdot \theta = \sum_{a \in I+} z_\theta(a)\,(l(a)\theta)\,,$$

where Z_θ is an operator on the ring $\hat{\mathcal{R}}$.

Z_θ can also be viewed as an element of a power series ring $\mathcal{P}$ in infinitely many variables $\{x_\pi\}$ indexed by the prime ideals π of K, with coefficients in the field of complex numbers. These power series operate on the vector space $\hat{\mathcal{R}} = \prod_\nu V_{k,\nu}(L)$ in such a way that the indeterminates x_π act as $l(\pi)$, that is, $x_\pi \cdot \theta = l(\pi)\theta$ for each prime ideal π of K. Indeed, if we let Z_θ be the following element of $\mathcal{P}$

$$Z_\theta = \sum_{a \in I+} z_\theta(a) x_a\,, \quad \text{where} \quad x_a = \prod_{\pi | a} x_\pi^{ord_\pi(a)},$$

then $Z_\theta \cdot \theta = \sum_a z_\theta(a) l(a)\theta$.

The main theorem of Shintani which we use is a characterization of those modular forms which are simultaneous eigenforms of the Hecke operators in terms of the system of coordinates $\{z_\theta(a) : \theta \in \mathcal{B}, a \in I^+\}$. Shintani's theorem asserts that a modular form $F \in A_k(L, \chi)$ is a simultaneous eigenform of the Hecke operators if and only if for every $\xi \in \Xi$ and every $\theta \in \mathcal{B}_\xi$ the formal Dirichlet series Z_θ is either identically zero or has the form

$$Z_\theta = z_\theta(1) Z_{\xi,\lambda},$$

where $Z_{\xi,\lambda}$ is an explicitly given formal Dirichlet series that depends only on ξ, χ and the eigenvalues λ of F. Moreover, $Z_{\xi,\lambda}$ admits an explicit Euler product factorization of the form:

$$Z_{\xi,\lambda} = \prod_p R_{\xi,\lambda,p},$$

where the Euler factors are explicit rational functions in the indeterminates $\{x_\pi : \pi | p\}$ and are given separately in three cases, depending on whether p is split, inert, or ramified in K.

This theorem of Shintani shows that in order to determine the adelic theta coefficients of a modular form F which is a simultaneous eigenform of the Hecke operators it is neccessary and sufficient to determine the coefficients $\{z_\theta(1) : \theta \in \mathcal{B}\}$ of the primitive components of the adelic theta coefficients and to determine the eigenvalues $\{\lambda(\pi)\}$ of F. This is precisely what is done in this paper for the Eisenstein series associated to unramified Hecke characters of the maximal torus. The method we use is an adaptation of the Siegel-Baily-Tsao-Karel technique for determining the Fourier coefficients of certain Eisenstein series on a tube domain.

Our main theorem asserts, more precisely, that if k is any positive integer and Λ is any unramified Hecke character of weight k on K, and if the Fourier-Jacobi expansion of the Eisenstein series E_Λ is

$$E_\Lambda = \sum_{r \in \mathbf{Q}} E'_{\Lambda,r},$$

where the Fourier-Jacobi coefficients are defined by the integrals

$$E'_{\Lambda,r} = \int_{N_{\mathbf{A}}} E_\Lambda(nm) \lambda(-r \cdot n) dn,$$

then each Fourier-Jacobi coefficient can be expressed as an infinite sum:

$$E'_{\Lambda,r} = \sum_{c \in \mathcal{N}(I+)} E'_{\Lambda,r} | M_{\mathbf{A}}(c),$$

where the sets $M_{\mathbf{A}}(c)$ are all mutually disjoint open sets whose union is all of $M_{\mathbf{A}}$, and each of the restrictions can be expressed as a finite sum:

$$E'_{\Lambda,r} | M_{\mathbf{A}}(c) = \sum_{\substack{(\nu,a) \\ \nu a \overline{a} = rc}} \sum_{\xi \in \Xi_\nu} L_\xi(k, \Lambda)\, z_\xi(a)\ (l(a)\theta_\xi)\, | M_{\mathbf{A}}(c),$$

where ν and a range over integral ideals of $\mathbf{Q}$ and K respectively, and where the theta function $\theta_\xi \in V_\xi$ and the complex numbers $L_\xi(k,\Lambda)$, $z_\xi(a)$ are given explicitly in terms k, ξ, a, and Λ. More precisely, the number $L_\xi(k,\Lambda)$ is a monomial of special values of Hecke and Dirichlet L-series and and the complex numbers $\{z_\xi(a) : a \in I^+\}$ are the coefficients of the formal powerseries

$$Z_\xi(k,\Lambda) = \sum_{a\in I^+} z_\xi(a) x_a,$$

which is given explicitly below as an Euler product.

The main formula for $E'_{\Lambda,r}$ just described is derived from a formula for the *adelic theta coefficients* $E_{\Lambda,\nu}$ of E_Λ:

$$E_{\Lambda,\nu} = \sum_{\substack{(\nu',a)\\ \nu'\mathcal{N}(a)=\nu}} \sum_{\xi\in\Xi_{\nu'}} L_\xi(k,\Lambda)\, z_\xi(a)\ (l(a)\theta_\xi)\,,$$

which can be expressed more elegantly when the adelic theta expansion of E_Λ is viewed as an element of $\hat{\mathcal{R}}$ as follows:

$$E_\Lambda = \sum_{\xi\in\mathcal{B}} L_\xi(k,\Lambda)\,(Z_\xi(k,\Lambda)\cdot\theta_\xi)\,.$$

The constant $L_\xi(k,\Lambda)$, the formal Dirichlet series $Z_\xi(k,\Lambda)$, and the theta function θ_ξ are given explicitly in terms of $\xi = (\nu, \kappa^*, \Sigma)$ as follows:

i) The adelic theta function θ_ξ is defined to be the element of $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$ dual (with respect to the Siegel inner product) to the linear functional l_Λ defined by

$$l_\Lambda(\theta) = \sum_{\beta\in CL_K} \Lambda(\beta_f)^{-1}\ e(-\nu\|\beta_f\|\sqrt{-D}/2)\ \theta(\beta_f),$$

where the summand is a well-defined function on the group $K^*\backslash K^*_{\mathbf{A}}/O^*_f K^*_\infty$ which is isomorphic to the class group CL_K. In other words, θ_ξ is the unique element in V_ξ such that $(\theta_\xi,\theta) = l_\Lambda(\theta)$ for all $\theta \in V_\xi$.

ii) the constant $L_\xi(k,\Lambda)$ is given explicitly as the following monomial of Hecke and Dirichlet L-series:

$$L_\xi(k,\Lambda) = \frac{(-1)^{3k/2}}{4\nu^2 D^{(3k+1)/2}}\ \alpha(k,\Lambda,\Sigma)\ \frac{L_K((3k-1)/2,\chi_2^*)}{L_{\mathbf{Q}}(3k-1,\chi_K)\ L_K(3k/2,\chi_1^*)}\,,$$

where

$$\alpha(k,\Lambda,\Sigma) = \prod_{\pi\in\Sigma}\left(1 + \Lambda^*_T(\pi)\mathcal{N}(\pi)^{-(3k/2)+1}\right),$$

and where χ_1^*, χ_2^* are Hecke characters of K (cf. §1.6) associated to the Hecke characters Λ and κ^* by

$$\chi_1^* = \overline{\Lambda_Z^2\Lambda_T^{*3}} \in \mathcal{H}^*_{-3k,1}(K), \qquad \chi_2^* = \overline{\Lambda_Z^2\Lambda_T^*}\kappa^* \in \mathcal{H}^*_{-(3k-1),C_\nu}(K).$$

iii) The formal Dirichlet series $Z_\xi(k, \Lambda)$ is given explicitly in §7.1. The formula there is derived using Shintani's general formula and our calculation of the eigenvalues of the Eisenstein series.

From this explicit formula, and an explicit isomorphism between the space of adelic theta functions and a space of classical theta functions, we easily deduce the arithmeticity of E_Λ using results of Siegel [14] on the special values of Dirichlet L-series and results of Damerell [4] on the special values of Hecke L-series.

Our formula is proved by extending ideas used by Siegel, Baily, and Tsao, to find the Fourier coefficients of modular forms on tube domains. The situation is, of course, more complicated here because the Fourier-Jacobi coefficients are sums of vectors of theta functions rather than complex numbers.

The first step in the proof is to use the Bruhat decomposition to express the Eisenstein series as a sum over $U_{\mathbf{Q}}$

$$E_\Lambda(g) = \Lambda(\delta(g)) + \sum_{u \in U_{\mathbf{Q}}} \Lambda(\delta(\iota u g)),$$

where ι is a representative of the nontrivial element in the Weyl group. From this representation, we can express the νth adelic theta coefficient of E_Λ (restricted to $M_{\mathbf{A}}$) in the form:

$$E_{\Lambda,\nu}(m) = \sum_{u \in N_{\mathbf{Q}} \backslash U_{\mathbf{Q}}} S_\Lambda(um, \nu \|\delta(m_f)\|),$$

where the summand is defined by an integral of the form

$$S_\Lambda(m, r) = \int_{N_{\mathbf{A}}} \Lambda(\delta(\iota n m))\lambda(-r \cdot n) dn,$$

and λ is the continuous character on $N_{\mathbf{Q}} \backslash N_{\mathbf{A}} / N(L)_f$ which is used to define the adelic theta coefficients.

The function S_Λ plays a major role in our work and we call it the Siegel function. In Chapter 5 we explicitly evaluate this function and show that, for each fixed $m \in M_{\mathbf{A}}$, it admits a factorization into Euler factors. This evaluation is done by explicitly calculating the Iwasawa map δ and using these calculations to explicitly evaluate the Siegel function. For example, if $m \in M_{\mathbf{A}}$ and $r \in \mathbf{Q}_{\mathbf{A}}$ then we show that

$$S_\Lambda(m, r) = \frac{(-2\pi i)^{3k}}{(3k-1)!} r_\infty^{3k-1} j_k(m_\infty, r_\infty) \prod_p S_{\Lambda,p}(m_p, r_p),$$

where $S_{\Lambda,p}(m_p, r_p) = (1 - p^{-3k})$ if r_p is a unit in $\mathbf{Z}_p$ and $m_p \in M(L)_p$. The restriction of the Siegel function to $M_f \times \mathbf{Z}_f^*$ does not however have compact support and this makes for interesting problems later on in the proof.

In Chapter 6, we show that the primitive component of $E_{\Lambda,\nu}$ is a finite sum

$$E^0_{\Lambda,\nu} = \sum_{\xi \in \Xi_\nu} L_\xi(k, \Lambda) \cdot \theta_\xi,$$

where θ_ξ is the element of V_ξ dual to the linear functional l_Λ defined above. Since the subspaces V_ξ are all orthogonal to each other, we can prove this formula by calculating the inner product of any $\theta \in V_\xi$ with $E_{\Lambda,\nu}$ and showing that

$$(E_{\Lambda,\nu}, \theta) = \overline{L_\xi(k, \Lambda)} l_\Lambda(\theta),$$

where the inner product is defined by the integral

$$(E_{\Lambda,\nu}, \theta) = \int_{M_{\mathbf{Q}} \backslash M_{\mathbf{A}}} \overline{E_{\Lambda,\nu}(m)} \theta(m) dm,$$

with respect to a suitably normalized Haar measure dm. Using the Siegel-Baily-Tsao-Karel form of the Eisenstein series, this inner product can be written in terms of the Siegel function as

$$(E_{\Lambda,\nu}, \theta) = \int_{N_{\mathbf{Q}} T_{\mathbf{Q}} \backslash M_{\mathbf{A}}} \overline{S_\Lambda(m, \nu)} \theta(m) dm.$$

The evaluation of this inner product is done in three steps.

In the first step, we evaluate the archimedean contribution to this integral and show that if θ is any theta function of level ν, then the inner product in question can be expressed as an integral that involves only the nonarchimedean values of the Siegel function and a transform of the theta function:

$$(E_{\Lambda,\nu}, \theta) = \epsilon_\infty \int_{J_K \times W_f} \mathcal{S}_{\Lambda,\theta}(a, w) da dw,$$

where the integrand,

$$\mathcal{S}_{\Lambda,\theta}(a, w) = \Lambda(a_f)^{-1} \overline{S_{\Lambda,f}(w, r a \overline{a} / \mathcal{N}(a))} \tilde{\theta}(aw),$$

involves the finite component of the Siegel function S_Λ and where J_K is the idele class group and $W_f \cong O_f$ is the quotient of the unipotent group U by its center, ϵ_∞ is an explicit algebraic number, and $\tilde{\theta}$ is a simple transform of the theta function.

In the rest of Chapter 6 we evaluate this integral by local methods. Since the integrand does not factor into a product of local functions, we must use a limiting argument. The main idea is to show that for every prime p there is a relation of the form

$$\int_{J_K \times W_p} \mathcal{S}_{\Lambda,\theta}(a, v + w_p) da dw_p = \epsilon_p \int_{J_K} \mathcal{S}_{\Lambda,\theta}(a, v) da,$$

where ϵ_p is an explicit Euler factor and v is any adele whose p-component is zero. To prove formulas of this form we need to assume that θ is in one of Shintani's eigenspaces $V_\xi = V_{k,\nu}(L)^0_{\kappa,\Sigma}$, and we need to use our explicit evaluation of the Siegel function.

Finally, in Chapter 7 we calculate the eigenvalues of the Eisenstein series and gather together the results of the previous Chapters into our main theorem. In Chapter 8, we deduce the arithmeticity of the Eisenstein series as an easy corollary of the main theorem.

ACKNOWLEDGEMENTS

I would like to thank all those who have given me support and encouragement in my graduate studies. Special thanks are due to my advisor, Walter Baily, Jr., for introducing me to modular forms on GU(2,1) and for always being ready to listen to my ideas and results. My second reader, Jon Rogawski, deserves thanks for showing me new ways to look at the problems I have considered herein. I would also like to thank David Zelinsky for his help in teaching me about Neron models and for helping out with the logistics of completing a thesis. Finally, I would like to thank Andee Rubin for her insightful and sympathetic encouragement.

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