Logic in Computer Science

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Introduction. Propositional logic

- History of logic
- Formal logic as a science
- The three worlds:
  - The real world
  - Proofs and theorems
  - Models and validity
- Propositional logic
  - Sentences
  - Natural deduction

Logic in Antiquity

The science of inferring correctly. Some conclusions only depend on the *form* of the argument and not on the actual contents.

Doesn't deal with *how* humans think (psychology) or whether the statements actually *agree* with facts (theory of knowledge).

Sokrates (approx. 469–399), Plato (427–347), Aristotle (384–322).

$$\frac{\text{All } M \text{ are } P \qquad \text{All } S \text{ are } M}{\text{All } S \text{ are } P} \qquad\qquad \frac{\text{All } P \text{ are } M \qquad \text{Some } S \text{ are not } M}{\text{Some } S \text{ are not } P}$$

Syllogisms

Four kinds of statements:

All/Some . . . are/are not . . .

two premises, a conclusion (256 "modes", 19 (15) valid ones).

Logic Systems

Formalism. A "game" with symbols	The actual (or imagined) world

Logic $\subseteq$ Philosophy, Logic $\subseteq$ Mathematics. **Circularity?** Mathematics used in logic, but logic used in (or even founding?) mathematical reasoning.

Desirable properties of a formal system:

sufficiency (expressibility): Has formulas for the items that interest us.

necessity : No superfluous symbols or notions.

consistency : Two contradictory statements never concluded.

soundness : Only true statements concluded.

completeness : All true statements concluded.

decidability : Checkable if a statement is concluded or not.

Modern Formal Logic

Gottfried Wilhelm Leibniz (1646–1716). George Boole (1815–64). Gottlob Frege (1848–1925). Guiseppe Peano (1858–1932). Bertrand Russell (1872–1970). Alfred North Whitehead (1861–1947).

Michael R.A. Huth, Mark D. Ryan, *Logic in Computer Science: Modelling and reasoning about systems*, Cambridge University Press 2000.

Symbols, formalism		World (object)
proof, theorem, deduction, $\vdash p$	model, consequence, validity, $\models p$, Alfred Tarski (1902–83) 1933	

premises : allegedproof $\rightarrow$ statements*
conclusion : allegedproof $\rightarrow$ statements
checkproof : allegedproof $\rightarrow$ `bool`

computable!

Propositional Logic

Judgements formed with *propositional variables* $p, q, r, p_1, \ldots$, and operators:

Negation $\overline{\phi}$, $-\phi$, $\neg\phi$
Disjunction Classically *exclusive* (lat. aut ... aut ...), now always *inclusive* (lat. vel ... vel ...), $\phi\ v\ \psi$, $\phi \vee \psi$, $\phi + \psi$
Conjunction $\phi \,\&\, \psi$, $\phi \cdot \psi$, $\phi\psi$, $\phi \wedge \psi$
Implication $\phi \supset \psi$, $\phi < \psi$, $\phi \Rightarrow \psi$, $\phi \rightarrow \psi$
Absurdity, contradiction 0, **F**, $\not{o}$, Λ, $\perp$ (bottom)

Priorities: $\neg$ binds tighter than $\{\vee, \wedge\}$, and they in turn bind tighter than $\rightarrow$. (Some texts let $\wedge$ bind tighter than $\vee$ (as $\cdot$ binds tighter than $+$), others let $\vee$ bind tighter than $\wedge$ (so that conjunctive normal form can be written completely without parentheses). We don't decide on a priority among them.)

Omitted: (for the moment)
Exclusive disjunction $+$, $\oplus$
Equivalence $=$, $\equiv$, $\Leftrightarrow$, $\leftrightarrow$
Tautology 1, **T**, $\bigvee$, $\top$

Natural Deduction

A *sequent* $\phi_1, \ldots, \phi_n \vdash \psi$

Gerhard Gentzen (1909–45)

The rules for conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \wedge\mathrm{i}$$

$$\frac{\phi \wedge \psi}{\phi} \wedge\mathrm{e}_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge\mathrm{e}_2$$

Example 1.4: $p \wedge q, r \vdash q \wedge r$

The rules of double negation

$$\frac{\neg\neg\phi}{\phi}\neg\neg\mathrm{e} \qquad \frac{\phi}{\neg\neg\phi}\neg\neg\mathrm{i}$$

(Later we shall see that the second rule can be derived from other rules.)

Example 1.5: $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$
Example 1.6: $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$

Implication

Modus ponens

$$\frac{\phi \qquad \phi \to \psi}{\psi} \to\text{e}$$

$p, p \to q, p \to (q \to r) \vdash r$

Modus tollens

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi} \text{MT}$$

Example 1.7: $p \to (q \to r), p, \neg r \vdash \neg q$

Example 1.8: $\neg p \to q, \neg q \vdash p$; $p \to \neg q, q \vdash \neg p$

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to\text{i}$$

Example 1.9: $\neg q \to \neg p \vdash p \to \neg\neg q$

$p \vdash p$;$\vdash p \to p$

Exm 1.11: $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

Example 1.13–1.14: $p \wedge q \to r \dashv\vdash p \to (q \to r)$

Example 1.15: $p \to q \vdash p \wedge r \to q \wedge r$

Disjunction

$$\frac{\phi}{\phi \vee \psi} \vee\mathrm{i}_1 \qquad \frac{\psi}{\phi \vee \psi} \vee\mathrm{i}_2$$

$$\frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \vee\mathrm{e}$$

$p \vee q \vdash q \vee p$

Example 1.16: $q \rightarrow r \vdash p \vee q \rightarrow p \vee r$

Example 1.17: $(p \vee q) \vee r \vdash p \vee (q \vee r)$

Example 1.18: $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

Negation

$$\frac{\bot}{\phi}\bot\mathrm{e}$$

$$\frac{\phi \qquad \neg\phi}{\bot}\neg\mathrm{e}$$

Example 1.20: $\neg p \vee q \vdash p \rightarrow q$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\neg\mathrm{i}$$

Exmple 1.21: $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$; $p \rightarrow \neg p \vdash \neg p$

Example 1.22: $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$

Example 1.23: $p \wedge \neg q \rightarrow r, \neg r, p \vdash q$

Derived rules

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\text{MT}$$

$$\frac{\phi}{\neg\neg\phi}\neg\neg\text{i}$$

Reductio ad absurdum

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\text{RAA}$$

Tertium non datur (LEM, *law of the excluded middle*)

$$\frac{}{\phi \vee \neg\phi}\text{TND}$$

Example 1.24: $p \to q \vdash \neg p \vee q$

Provable equivalence

$$\neg(p \wedge q) \dashv\vdash \neg p \vee \neg q$$
$$\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$$
$$p \rightarrow q \dashv\vdash \neg q \rightarrow \neg p$$
$$p \rightarrow q \dashv\vdash \neg p \vee q$$
$$p \wedge q \rightarrow p \dashv\vdash r \vee \neg r$$
$$p \wedge q \rightarrow r \dashv\vdash p \rightarrow (q \rightarrow r)$$

Intuitionistic Logic

Luitzen Egbertus Jan Brouwer (1881–1966)

Intuitionists claim RAA, TND and $\neg\neg$e invalid.

Theorem 1.26: *There exist positive irrational numbers a and b such that a^b is a rational number.*

Proof (not intuitionistically valid): Choose $a = b = \sqrt{2}$, if $\sqrt{2}^{\sqrt{2}}$ is rational, and $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$ otherwise.