

Natural Deduction for Propositional Logic

James Pustejovsky



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5.1 Natural deduction rules for $\wedge, \rightarrow, \vee$

There are two rules for each connective. The rules reflect the meanings of the connectives.

The easiest is \wedge ('and').

Rules for \wedge

- (\wedge -introduction, or $\wedge I$) To introduce a formula of the form $A \wedge B$, you have to have already introduced A and B .

1	A	we proved this...
	\vdots	(other junk)
2	B	and this...
3	$A \wedge B$	$\wedge I(1, 2)$

The line numbers are essential for clarity.

Rules for \wedge ctd.

- (\wedge -elimination, or $\wedge E$) If you have managed to write down $A \wedge B$, you can go on to write down A and/or B .

1 $A \wedge B$ we proved this somehow
2 A $\wedge E(1)$
3 B $\wedge E(1)$

Rules for \vee

- (\vee -introduction, or $\vee I$)
To prove $A \vee B$, prove A , or (if you prefer) prove B .

1 A proved this somehow
2 $A \vee B$ $\vee I(1)$

B can be any formula at all!

1 B proved this somehow
2 $A \vee B$ $\vee I(1)$

A can be any formula at all.

Rules for \vee , ctd.

- (\vee -elimination, or $\vee E$) To prove something from $A \vee B$, you have to prove it by assuming A , AND prove it by assuming B . (This is arguing by cases.)

1	$A \vee B$		we got this somehow
2	A	ass	
3	:	the 1st proof	
4	C	we got it	
5	B	ass	
6	:	the 2nd proof	
7	C	we got it again	
8	C		$\vee E(1, 2, 4, 5, 7)$

The assumptions A, B are not usable later, so are put in (side-by-side) boxes. *Nothing inside the boxes can be used later.*

Rules for \rightarrow

- (\rightarrow -introduction, $\rightarrow I$: 'arrow-introduction') To introduce a formula of the form $A \rightarrow B$, you *assume* A and then prove B . During the proof, you can use A as well as anything already established. But *you can't use A or anything from the proof of B from A later on* (because it was based on an extra assumption). So we isolate the proof of B from A , in a *box*:

1	A	ass
	(the proof)	hard struggle
2	B	we made it!
3	$A \rightarrow B$	$\rightarrow I(1, 2)$

Nothing inside the box can be used later.

In natural deduction, boxes are used when we make additional assumptions. The first line inside a box should always be labelled 'ass' (assumption) — with one exception, coming later (p. 212).

Rules for \rightarrow , ctd.

- (\rightarrow -elimination, or $\rightarrow E$) If you have managed to write down A and $A \rightarrow B$, in either order, you can go on to write down B . (This is modus ponens.)

1	$A \rightarrow B$	we got this somehow...
	\vdots	other junk
2	A	and this too...
3	B	$\rightarrow E(1, 2)$

5.3 Rules for \neg

This is the trickiest case. Also, \neg has three rules! The first two treat $\neg A$ like $A \rightarrow \perp$.

- (\neg -introduction, $\neg I$) To prove $\neg A$, you assume A and prove \perp . As usual, you can't then use A later on, so enclose the proof of \perp from assumption A in a box:

1	A	ass
2	\vdots	more hard work, oh no
3	\perp	we got it!
4	$\neg A$	$\neg I(1, 3)$

Rules for \neg , ctd.

- (\neg -elimination, $\neg E$)

From A and $\neg A$, deduce \perp :

1	$\neg A$	proved this somehow...
2	\vdots	junk
3	A	... and this
4	\perp	$\neg E(1, 3)$

- ($\neg\neg$ -elimination, $\neg\neg$):

From $\neg\neg A$, deduce A . (See example 5.8.)