Logic, Human Logic, and Propositional Logic

Foundations of Semantics

LING 130

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Human Logic

Fragments of Information

The red block is on the green block.
*The green block is somewhere **above** the blue block.*
*The green block is **not** on the blue block.*
*The yellow block is on the green block **or** the blue block.*
*There is **some** block on the black block.*

A block can be on only one other block or the table (not both).
A block can have at most one block on top.
There are exactly 5 blocks.

Conclusions

The red block is on the green block.
The green block is on the yellow block.
The yellow block is on the blue block.
The blue block is on the black block.
The black block is directly on the table.

Proof

The yellow block is on the green block or the blue block.
The red block is on the green block.
A block can have at most one block on top.
Therefore, the yellow block is not on the green block.
Therefore, the yellow block must be on the blue block.

Reasoning by Pattern

All Accords are Hondas.
All Hondas are Japanese.
Therefore, all Accords are Japanese.

All borogoves are slithy toves.
All slithy toves are mimsy.
Therefore, all borogoves are mimsy.

All x are y.
All y are z.
Therefore, all x are z.

Questions

Which patterns are correct?

How many patterns are enough?

Unsound Patterns

Pattern

All x are y.
Some y are z.
Therefore, some x are z.

Good Instance

All Toyotas are Japanese cars.
Some Japanese cars are made in America.
Therefore, some Toyotas are made in America.

Not-So-Good Instance

All Toyotas are cars.
Some cars are Porsches.
Therefore, some Toyotas are Porsches.

Induction - Unsound

I have seen 1000 black ravens.
I have never seen a raven that is not black.
Therefore, every raven is black.
Now try red Hondas.

Abduction - Unsound

If there is no fuel, the car will not start.
If there is no spark, the car will not start.
There is spark.
The car will not start.
Therefore, there is no fuel.
What if the car is in a vacuum chamber?

Deduction - Sound

Logical Entailment/Deduction:
- Does not say that conclusion is true in general
- Conclusion true *whenever* premises are true

Leibniz: *The intellect is freed of all conception of the objects involved, and yet the computation yields the correct result.*

Russell: *Math may be defined as the subject in which we never know what we are talking about nor whether what we are saying is true in the world.*

Formal Logic

Formal Mathematics

Algebra
1. Formal language for encoding information
2. Legal transformations

Logic
1. Formal language for encoding information
2. Legal transformations

Algebra Problem

Sophia is three times as old as Sasha. Sophia's age and Sasha's age add up to twelve. How old are Sophia and Sasha?

$$\begin{array}{r} x - 3y = 0 \\ \underline{x + y = 12} \\ -4y = -12 \end{array}$$

$$y = 3$$

$$x = 9$$

Logic Problem

If Mary loves Pat, then Mary loves Quincy. If it is Monday, then Mary loves Pat or Quincy. If it is Monday, does Mary love Quincy?

If it is Monday, does Mary love Pat?

Mary loves only one person at a time. If it is Monday, does Mary love Pat?

Formalization

Simple Sentences:

Mary loves Pat.	p
Mary loves Quincy.	q
It is Monday.	m

Premises:

If Mary loves pat, Mary loves Quincy.	$p \Rightarrow q$
If it Monday, Mary loves Pat or Quincy.	$m \Rightarrow p \vee q$
Mary loves one person at a time.	$p \wedge q \Rightarrow$

Questions:

Does Mary love Pat?	$\Rightarrow p$
Does Mary love Qunicy?	$\Rightarrow q$

Rule of Inference

Propositional Resolution

$$\frac{\begin{array}{ccc} p_1 \wedge \ldots \wedge p_k & \Rightarrow & q_1 \vee \ldots \vee q_l \\ r_1 \wedge \ldots \wedge r_m & \Rightarrow & s_1 \vee \ldots \vee s_n \end{array}}{p_1 \wedge \ldots \wedge p_k \wedge r_1 \wedge \ldots \wedge r_m \;\Rightarrow\; q_1 \vee \ldots \vee q_l \vee s_1 \vee \ldots \vee s_n}$$

NB: If p_i on the left hand side of one sentence is the same as q_j in the right hand side of the other sentence, it is okay to drop the two symbols, with the proviso that *only one* such pair may be dropped.

NB: If a constant is repeated on the same side of a single sentence, all but one of the occurrences can be deleted.

Examples

$$\frac{\begin{array}{ccc} p & \Rightarrow & q \\ & \Rightarrow & p \end{array}}{\begin{array}{ccc} & \Rightarrow & q \end{array}} \qquad \frac{\begin{array}{ccc} p & \Rightarrow & q \\ q & \Rightarrow & \end{array}}{\begin{array}{ccc} p & \Rightarrow & \end{array}} \qquad \frac{\begin{array}{ccc} p & \Rightarrow & q \\ q & \Rightarrow & r \end{array}}{\begin{array}{ccc} p & \Rightarrow & r \end{array}}$$

Logic Problem Revisited

If Mary loves Pat, then Mary loves Quincy. If it is Monday, then Mary loves Pat or Quincy. If it is Monday, does Mary love Quincy?

$$\frac{\begin{array}{ccc} p & \Rightarrow & q \\ m & \Rightarrow & p \vee q \end{array}}{\begin{array}{ccc} m & \Rightarrow & q \vee q \\ m & \Rightarrow & q \end{array}}$$

Logic Problem Concluded

Mary loves only one person at a time. If it is Monday, does Mary love Pat?

$$\frac{\begin{array}{ccc} m & \Rightarrow & q \\ p \wedge q & \Rightarrow & \end{array}}{\begin{array}{ccc} m \wedge p & \Rightarrow & \end{array}}$$

Compound Sentences

Negations:

$$\neg raining$$

The argument of a negation is called the *target*.

Conjunctions:

$$(raining \wedge snowing)$$

The arguments of a conjunction are called *conjuncts*.

Disjunctions:

$$(raining \vee snowing)$$

The arguments of a disjunction are called *disjuncts*.

Compound Sentences (concluded)

Implications:

$$(raining \Rightarrow cloudy)$$

The left argument of an implication is the *antecedent*.
The right argument of an implication is the *consequent*.

Reductions:

$$(cloudy \Leftarrow raining)$$

The left argument of a reduction is the *consequent*.
The right argument of a reduction is the *antecedent*.

Equivalences:

$$(cloudy \Leftrightarrow raining)$$

Parenthesis Removal

Dropping Parentheses is good:

$$(p \wedge q) \rightarrow p \wedge q$$

But it can lead to ambiguities:

$$((p \vee q) \wedge r) \rightarrow p \wedge q \vee r$$
$$(p \vee (q \wedge r)) \rightarrow p \wedge q \vee r$$

Precedence

Parentheses can be dropped when the structure of an expression can be determined on the basis of precedence.

$$\neg$$
$$\wedge$$
$$\vee$$
$$\Rightarrow \Leftarrow \Leftrightarrow$$

NB: An operand associates with operator of higher precedence. If surrounded by operators of equal precedence, the operand associates with the operator to the right.

$p \wedge q \vee r$	$p \Rightarrow q \Rightarrow r$	$\neg p \wedge q$
$p \vee q \wedge r$	$p \Rightarrow q \Leftarrow r$	

Propositional Logic Interpretation

A *propositional logic interpretation* is an association between the propositional constants in a propositional language and the truth values T or F.

$p \xrightarrow{i} T$ $p^i = T$

$q \xrightarrow{i} F$ $q^i = F$

$r \xrightarrow{i} T$ $r^i = T$

The notion of interpretation can be extended to all sentences by application of operator semantics.

Operator Semantics

Negation:

ϕ	$\neg\phi$
T	F
F	T

For example, if the interpretation of p is F, then the interpretation of $\neg p$ is T.

For example, if the interpretation of $(p \wedge q)$ is T, then the interpretation of $\neg(p \wedge q)$ is F.

Operator Semantics (continued)

Conjunction:

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

NB: The semantics of disjunction here is often called *inclusive or*, which says that a disjunction is true if and only if *at least* one of its disjuncts is true. This is in contrast with *exclusive or*, according to which a disjunction is true if and only if an odd number of its disjuncts is true. What is the truth table for exclusive or?

Operator Semantics (continued)

Implication:

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Reduction:

ϕ	ψ	$\phi \Leftarrow \psi$
T	T	T
T	F	T
F	T	F
F	F	T

NB: The semantics of implication here is called *material implication*. It has the peculiar characteristic that any implication is true if the antecedent is false, whether or not there is a connection to the consequent. For example, the following is a true sentence.

If George Washington is alive, I am a billionaire.

Operator Semantics (concluded)

Equivalence:

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Evaluation

Interpretation i:

$$p^i = T$$
$$q^i = F$$
$$r^i = T$$

Compound Sentence

$$(p \vee q) \wedge (\neg q \vee r)$$

Multiple Interpretations

Logic does not prescribe which interpretation is "correct". In the absence of additional information, one interpretation is as good as another.

Interpretation i

$$p^i = T$$
$$q^i = F$$
$$r^i = T$$

Interpretation j

$$p^j = F$$
$$q^j = F$$
$$r^j = T$$

Examples:
- Different days of the week
- Different locations
- Beliefs of different people

Truth Tables

A *truth table* is a table of all possible interpretations for the propositional constants in a language.

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

One column per constant.

One row per interpretation.

For a language with n constants, there are 2^n interpretations.

Evaluation and Disambiguation

Evaluation:

$$p^i = T \qquad (p \vee q)^i = T$$
$$q^i = F \quad \longrightarrow \quad (\neg q)^i = T$$

Disambiguation:

$$(p \vee q)^i = T \qquad p^i = T$$
$$(\neg q)^i = T \quad \longrightarrow \quad q^i = F$$

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

p	*q*	*r*
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

$q \Rightarrow r$

p	*q*	*r*	
1	1	1	
1	1	0	×
1	0	1	
1	0	0	
0	1	1	
0	1	0	×
0	0	1	
0	0	0	

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

$q \Rightarrow r$

$p \Rightarrow q \wedge r$

p	*q*	*r*	
1	1	1	
1	1	0	×
1	0	1	×
1	0	0	×
0	1	1	
0	1	0	×
0	0	1	
0	0	0	

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

$q \Rightarrow r$

$p \Rightarrow q \wedge r$

$\neg r$

p	q	r	
1	1	1	×
1	1	0	×
1	0	1	×
1	0	0	×
0	1	1	×
0	1	0	×
0	0	1	×
0	0	0	

Properties of Sentences

Valid
Contingent
Unsatisfiable

A sentence is *valid* if and only if *every* interpretation satisfies it.

A sentence is *contingent* if and only if *some* interpretation satisfies it and *some* interpretation falsifies it.

A sentence is *unsatisfiable* if and only if *no* interpretation satisfies it.

Example of Validity

p	q	r	$(p \Rightarrow q)$	$(q \Rightarrow r)$	$(p \Rightarrow q) \vee (q \Rightarrow r)$
1	1	1			
1	1	0			
1	0	1			
1	0	0			
0	1	1			
0	1	0			
0	0	1			
0	0	0			

More Validities

Double Negation:

$$p \Leftrightarrow \neg\neg p$$

deMorgan's Laws:

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$

$$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$

Implication Introduction:

$$p \Rightarrow (q \Rightarrow p)$$

Implication Distribution

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Deduction

In deduction, the conclusion is true whenever the premises are true.

Premise: p
Conclusion: $(p \vee q)$

Premise: p
Non-Conclusion: $(p \wedge q)$

Premises: p, q
Conclusion: $(p \wedge q)$

Logical Entailment

A set of premises Δ *logically entails* a conclusion φ (written as $\Delta \models \varphi$) if and only if every interpretation that satisfies the premises also satisfies the conclusion.

$$\{p\} \models (p \vee q)$$

$$\{p\} \not\models (p \wedge q)$$

$$\{p,q\} \models (p \wedge q)$$

Truth Table Method

We can check for logical entailment by comparing tables of all possible interpretations.

In the first table, eliminate all rows that do not satisfy premises.

In the second table, eliminate all rows that do not satisfy the conclusion.

If the remaining rows in the first table are a subset of the remaining rows in the second table, then the premises logically entail the conclusion.

Example

Does p logically entail $(p \vee q)$?

p	q
1	1
1	0
0	1
0	0

p	q
1	1
1	0
0	1
0	0

Example

Does p logically entail $(p \wedge q)$?

p	q
1	1
1	0
0	1
0	0

p	q
1	1
1	0
0	1
0	0

Does $\{p,q\}$ logically entail $(p \wedge q)$?

Example

If Mary loves Pat, then Mary loves Quincy.
If it is Monday, then Mary loves Pat or Quincy.
If it is Monday, does Mary love Pat?

m	*p*	*q*
1	1	1
×	×	×
1	0	1
×	×	×
0	1	1
×	×	×
0	0	1
0	0	0

m	*p*	*q*
1	1	1
×	×	×
1	0	1
×	×	×
0	1	1
0	1	0
0	0	1
0	0	0

Problem

There can be many, many interpretations for a Propositional Language.

Remember that, for a language with n constants, there are 2^n possible interpretations.

Sometimes there are many constants among premises that are irrelevant to the conclusion. Much wasted work.

Answer: Proofs

Patterns

A *pattern* is a parameterized expression, i.e. an expression satisfying the grammatical rules of our language except for the occurrence of meta-variables (Greek letters) in place of various subparts of the expression.

Sample Pattern:

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

Instance:

$$p \Rightarrow (q \Rightarrow p)$$

Instance:

$$(p \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Rules of Inference

A *rule of inference* is a rule of reasoning consisting of one set of sentence patterns, called *premises*, and a second set of sentence patterns, called *conclusions*.

$$\frac{\varphi \Rightarrow \psi \quad \varphi}{\psi}$$

Rule Instances

An *instance* of a rule of inference is a rule in which all meta-variables have been consistently replaced by expressions in such a way that all premises and conclusions are syntactically legal sentences.

$$\frac{raining \Rightarrow wet \quad raining}{wet} \qquad \frac{wet \Rightarrow slippery \quad wet}{slippery}$$

$$\frac{p \Rightarrow (q \Rightarrow r) \quad p}{q \Rightarrow r} \qquad \frac{(p \Rightarrow q) \Rightarrow r \quad p \Rightarrow q}{r}$$

Sound Rules of Inference

A rule of inference is *sound* if and only if the premises in any instance of the rule logically entail the conclusions.

Modus Ponens (MP)

$$\frac{\varphi \Rightarrow \psi \quad \varphi}{\psi}$$

Modus Tolens (MT)

$$\frac{\varphi \Rightarrow \psi \quad \neg\psi}{\neg\varphi}$$

Equivalence Elimination (EE)

$$\frac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi \quad \psi \Rightarrow \varphi}$$

Double Negation (DN)

$$\frac{\neg\neg\varphi}{\varphi}$$

Proof (Version 1)

A *proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either:

1. a premise
2. the result of applying a rule of inference to earlier items in sequence.

Example

When it is raining, the ground is wet. When the ground is wet, it is slippery. It is raining. Prove that it is slippery.

1.	$raining \Rightarrow wet$	Premise
2.	$wet \Rightarrow slippery$	Premise
3.	$raining$	Premise
4.	wet	MP : 1,3
5.	$slippery$	MP : 2,4

Error

Note: Rules of inference apply only to top-level sentences in a proof. Sometimes works but sometimes fails.

	1.	$raining \Rightarrow cloudy$	Premise	
No!	2.	$raining \Rightarrow wet$	Premise	No!
	3.	$cloudy \Rightarrow wet$	MP : 1,2	

Example

Heads you win. Tails I lose. Suppose the coin comes up tails. Show that you win.

Axiom Schemata

Fact: If a sentence is valid, then it is true under all interpretations. Consequently, there should be a proof without making any assumptions at all.

Fact: $(p \Rightarrow (q \Rightarrow p))$ is a valid sentence.

Problem: Prove $(p \Rightarrow (q \Rightarrow p))$.

Solution: We need some rules of inference without premises to get started.

An *axiom schema* is sentence pattern construed as a rule of inference without premises.

Rules and Schemata

Axiom Schemata as Rules of Inference

$$\varphi \Rightarrow (\psi \Rightarrow \varphi) \qquad \frac{}{\varphi \Rightarrow (\psi \Rightarrow \varphi)}$$

Rules of Inference as Axiom Schemata

$$\frac{\varphi \Rightarrow \psi \quad \neg\psi}{\neg\varphi} \qquad (\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)$$

Note: Of course, we must keep a least one rule of inference to use the schemata. By convention, we retain Modus Ponens.

Valid Axiom Schemata

A *valid axiom schema* is a sentence pattern denoting an infinite set of sentences, all of which are valid.

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

Standard Axiom Schemata

II: $\varphi \Rightarrow (\psi \Rightarrow \varphi)$

ID: $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

CR: $(\neg\psi \Rightarrow \varphi) \Rightarrow ((\neg\psi \Rightarrow \neg\varphi) \Rightarrow \psi)$
$(\psi \Rightarrow \varphi) \Rightarrow ((\psi \Rightarrow \neg\varphi) \Rightarrow \neg\psi)$

EQ: $(\varphi \Leftrightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$
$(\varphi \Leftrightarrow \psi) \Rightarrow (\psi \Rightarrow \varphi)$
$(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \varphi) \Rightarrow (\varphi \Leftrightarrow \psi))$

OQ: $(\varphi \Leftarrow \psi) \Leftrightarrow (\psi \Rightarrow \varphi)$
$(\varphi \vee \psi) \Leftrightarrow (\neg\varphi \Rightarrow \psi)$
$(\varphi \wedge \psi) \Leftrightarrow \neg(\neg\varphi \vee \neg\psi)$

Sample Proof

Whenever p is true, q is true. Whenever q is true, r is true. Prove that, whenever p is true, r is true.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	II
4.	$p \Rightarrow (q \Rightarrow r)$	MP : 3,2
5.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	MP : 5,4
7.	$p \Rightarrow r$	MP : 6,1

Proof (Official Version)

A *proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either:

1. a premise
2. An instance of an axiom schema
3. the result of applying a rule of inference to earlier items in sequence.

Provability

A conclusion is said to be *provable* from a set of premises (written $\Delta \vdash \varphi$) if and only if there is a finite proof of the conclusion from the premises using only Modus Ponens and the Standard Axiom Schemata.

Soundness and Completeness

Soundness: Our proof system is *sound*, i.e. if the conclusion is provable from the premises, then the premises propositionally entail the conclusion.

$$(\Delta \vdash \varphi) \Rightarrow (\Delta \models \varphi)$$

Completeness: Our proof system is *complete*, i.e. if the premises propositionally entail the conclusion, then the conclusion is provable from the premises.

$$(\Delta \models \varphi) \Rightarrow (\Delta \vdash \varphi)$$

Truth Tables and Proofs

The truth table method and the proof method succeed in exactly the same cases.

On large problems, the proof method often takes fewer steps than the truth table method. However, in the worst case, the proof method may take just as many or more steps to find an answer as the truth table method.

Usually, proofs are much smaller than the corresponding truth tables. So writing an argument to convince others does not take as much space.

Metatheorems

Deduction Theorem: Δ |- (φ ⇒ ψ) if and only if Δ∪{φ} |- ψ.

Equivalence Theorem: Δ |- (φ ⇔ ψ) and Δ |- χ, then it is the case that Δ |- $\chi_{\varphi \leftarrow \psi}$.

Proof Using Deduction Theorem

Problem: {$p \Rightarrow q$, $q \Rightarrow r$} |- ($p \Rightarrow r$)?

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	p	Premise
4.	q	MP : 1,3
5.	r	MP : 2,4

Proof Without Deduction Theorem

Problem: {$p \Rightarrow q$, $q \Rightarrow r$} |- ($p \Rightarrow r$)?

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	II
4.	$p \Rightarrow (q \Rightarrow r)$	MP : 3,2
5.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	MP : 5,4
7.	$p \Rightarrow r$	MP : 6,1