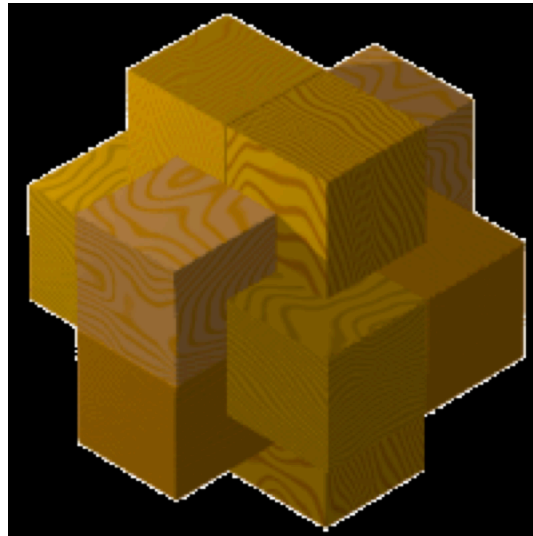




(from <http://www.research.ibm.com/BurrPuzzles/index.html>)

## The burr puzzles site



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## Background

Little is known about the early history of interlocking puzzles, but they were certainly produced both in Asia and Europe as early as the 18th century. Six-piece burrs were shown as early as 1803 in the Bestelmeier Toy catalogs. But it wasn't until Edwin Wyatt published *Puzzles in Wood* in 1928 that a book devoted to many interlocking puzzles was available. Puzzles of this type used to be known as "Chinese" puzzles, probably because they were mass-produced in the Orient since the early 1900s, but there does not appear to be any evidence that the idea originated there. Nowadays they are commonly referred to as "burr" puzzles. Wyatt introduced the term because the puzzles looked like a seed burr.

By far the most familiar burr is the six-piece burr. The six-piece burr is actually a large family of designs, since the designer has a wide choice of how to notch each piece. Several versions have been patented and manufactured. The earliest US patent is No. 1,225,760 of Brown, dated 1917, with several others following shortly thereafter.

Most toy and novelty stores have a few burr puzzles on their shelves. Unfortunately these are often the uninspired, time-worn versions with sliding keys and internal symmetries. The puzzle has suffered from this tarnished image, and to make matters worse, inventors have tinkered with bizarre embellishments like cords through holes to give the basic puzzle their twist.

In 1978, Bill Cutler has returned to the basic design and published a paper describing the solid six-piece burr completely [Cutler78]. He has shown that there are 25 possible notchable pieces to make solid six-piece burrs and that they can be put together in 314 ways. He also showed that there are 369 general pieces usable to make solid burrs and that they can be put together in 119,979 ways.

More recently, Stewart Coffin, Bill Cutler, Philippe Dubois, and Peter Marineau have come up with higher-level six-piece burr designs that are not solid and in which several pieces must be moved before one can be taken out. Stewart Coffin has enhanced the art of burr designs into abstract geometrical forms. His book *The Puzzling World of Polyhedral Dissections* contains close to 100 exquisite designs of burr and other wooden puzzles.

From the late 1980s to the mid 1990 Bill Cutler and others undertook a complete analysis of all six-piece burrs [Cutler94]. Original estimates by Cutler assumed it would take about 62.5 years to run the analysis on a PC AT. With some help from others who run Cutlers programs on faster and bigger machines, total calculation time was cut down to 2.5 years. From this analysis we now know that there are roughly 35.65 billion ways to assemble burr puzzles pieces (71.3 billion if mirror images are counted also). Of these 35.65 billion logical assemblies 5.95 billion can be taken apart. The highest level found was a level 12 puzzle. Unfortunately it is not unique, i.e. it has more than one assembly, most of which are of a much lower level. The highest level unique six-piece burr is of level 10 if the pieces are 8 units long and level 9 if the pieces are 6 units in length. If all pieces are notchable, the highest level is 5 for a unique burr.

## Six Piece Burrs

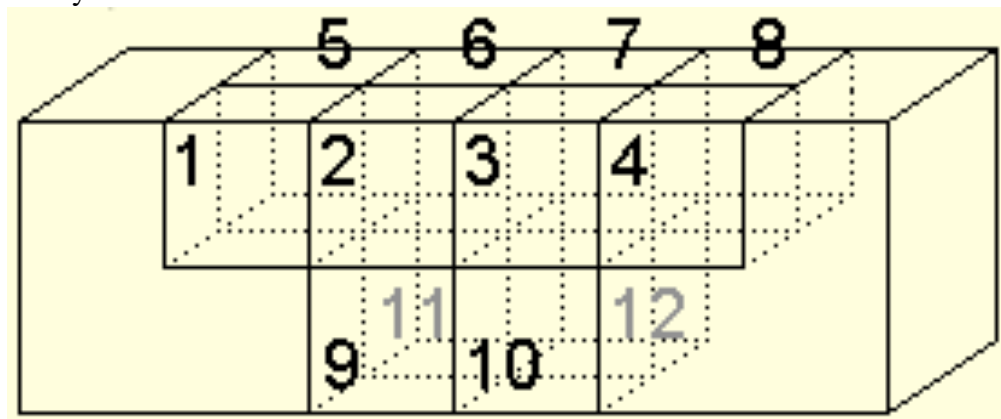
Six-piece burrs are the simplest regular burrs. To analyze the six-piece burrs, we first discuss the pieces used to build them. Then we introduce a few interesting facts about six-piece burrs. Finally we present some statistics from a (mostly) complete analysis made by Bill Cutler.

### Six-piece burr pieces

Before an analysis of six-piece burrs can be conducted, the usable pieces need to be computed. William Cutler was the first to completely analyze six-piece burr pieces used to make solid six-piece burrs [Cutler78].

To get a complete analysis we need to find a way to enumerate the pieces. To be able to refer to particular pieces, we introduce the following numbering scheme:

**Def 9.0:** Binary Piece Number



As we have seen in Lemma 2.1.2.2 a six-piece burr piece may have only the cubes 1 through 12 notched out. If we write down a 1 or 0 for the presence or absence of any of the twelve cubes we get the binary number of a piece.

**Lemma 9.0.1:** The solid six-piece burr piece has a binary number of 111111111111.

*Proof:* The solid piece does not have any cubes removed, hence all of the twelve cubes are present.

**Lemma 9.0.2:** The lightest usable piece has the binary number 000000000011.

*Proof:* The following image shows the cubes 1 through 10 are absent and 10 through 12 are present. Removing any more cubes would leave the piece disjoint.

**Def 9.1:** Decimal Piece Number

The decimal piece number is defined as 4096 minus the binary piece number, expressed as a decimal number as shown below:

$$\begin{array}{r} 4096 - 111111111111\text{ b} \\ | | | | | | | | | | | | | | | | | + - * 2048 \\ | | | | | | | | | | | | | | | | | + - - * 1024 + \\ | | | | | | | | | | | | | | | | | + - - - * 512 + \\ | | | | | | | | | | | | | | | | | + - - - - * 256 + \\ | | | | | | | | | | | | | | | | | + - - - - - * 128 + \\ | | | | | | | | | | | | | | | | | + - - - - - - * 64 + \\ | | | | | | | | | | | | | | | | | + - - - - - - - * 32 + \\ | | | | | | | | | | | | | | | | | + - - - - - - - - * 16 + \\ | | | | | | | | | | | | | | | | | + - - - - - - - - - * 8 + \\ | | | | | | | | | | | | | | | | | + - - - - - - - - - - * 4 + \\ | | | | | | | | | | | | | | | | | + - - - - - - - - - - - * 2 + \\ + - - - - - - - - - - - - - - - - - * 1 + \end{array}$$

*Note:* the reason for this seemingly reverse definition from normal mathematical notations of binary numbers was chosen for a graphical reason. If a program lists all the pieces sorted by decimal numbers, then this definition will yield generally better visible 3D representations of the pieces.

**Lemma 9.1.1:** The solid six-piece burr piece has the decimal number 1.

*Proof:*  $4096 - 111111111111\text{b} = 4096 - 4095 = 1.$

*Note:* The fact that the solid piece comes out as number 1 is the main reasons for choosing to subtract the binary number from 4096.

**Lemma 9.1.2:** The lightest usable piece has the decimal number 1024.

*Proof:*  $4096 - 00000000011\text{b} = 4096 - 2048 - 1024 = 1024.$

*Note:* If you rotate the piece by 180o you get the piece 000000001100 which has the decimal number  $4096 - 000000001100\text{b} = 4096 - 512 - 128 = 3456.$

We have already introduced notchable pieces in Def 1.2.1.1.2. now we are going to define the type of a six-piece burr piece with a finer grain of detail.

**Def 9.2:** internal corner

A piece is said to have an internal corner, if the sides of three cubes meet inside the piece in a concave fashion. The piece shown to the right has one internal corner.

**Lemma 9.2.1:** The maximum number of internal corners a piece can have is 8.

*Proof:* The piece on the right shows 8 internal corners, trying to remove any more cubes e.g from the bottom either reduces the number of internal corners or makes the piece disjoint.

**Def 9.3:** millable

A piece is called millable if it can be produced by a milling machine. This means the piece may not have any internal corners. For example the piece shown in the picture is millable.

**Lemma 9.3.1:** Notchable pieces are millable.

*Proof:* Since notchable pieces have no concave crosscuts perpendicular to the axis, no internal corners may be formed. Hence they are millable.

**Lemma 9.3.2:** There are millable pieces which are not notchable.

*Proof:* The piece shown in the definition 9.3 above, is millable but not notchable.

#### **Def 9.4:** Piece Type

Piece types are introduced as to whether a piece is notchable, millable, or neither. And within those three categories they are divided as to whether they can be used to build solvable, solid six-piece burrs.

#### **Type Description**

notchable32

notchable piece that can be used for solid burrs

notchable

notchable piece that can be used for general burrs

millable32

millable piece that can be used for solid burrs

millable

millable piece that can be used for general burrs

general32

general piece that can be used for solid burrs

general

general piece that can be used for general burrs

disjoint

unusable piece

*Note:* the numeral 32 signifies the fact that a solid burr has a weight of 32 (see Lemma 4.1.2).

**Theorem 9.4.1:** A notched piece is usable for solid burrs (of type notched32) if it meets the following criteria:

1. The piece is solid
2. The following cubes may only be removed as a pair: 1 and 5, 4 and 8, 9 and 10, 11 and 12. Cubes 2, 3, 6, and 7 may be removed individually.

*Proof:* see [Cutler78].

**Theorem 9.4.2:** A general piece is usable for solid burrs (of type general32) if it meets the following criteria:

1. The piece is solid
2. Cubes 1, 2, 5 and 6 removed; others may vary
3. Cubes 3, 4, 7 and 8 removed; others may vary
4. Cubes 2, 3, 9 and 10 removed
5. Cubes 6, 7, 11 and 12 removed
6. Cubes 1 and 5 removed and either (a) 3 and 7 removed, (b) 4 and 8 removed, or (c) 4 and 8 not removed
7. Cubes 2 and 6 removed and either (a) 4 and 8 removed, or (b) 4 and 8 not removed
8. Cubes 1 and 5 not removed and (a) 3 and 7 removed, or (b) 4 and 8 removed

*Proof:* see [Cutler78].

**Theorem 9.4.3:** For each type there exist the following number of usable, unique pieces. That is duplicates (from rotations) have been removed.

notchable32

25

<http://www.research.ibm.com/BurrPuzzles/Burr6Pieces25C.html> (20KB)

notchable

59

<http://www.research.ibm.com/BurrPuzzles/Burr6Pieces59C.html> (44KB)

millable32

78

<http://www.research.ibm.com/BurrPuzzles/Burr6Pieces78C.html> (57KB)

general32

369

<http://www.research.ibm.com/BurrPuzzles/Burr6Pieces369C.html> (250KB) takes long to load

general

837

<http://www.research.ibm.com/BurrPuzzles/Burr6Pieces837C.html> (554KB) takes forever to load

*Proof:* enumeration by computer programs, using Theorems 9.4.1 and 9.4.2.

# Burr Weight

## **Def 9.5:** Piece Weight

The weight of a six-piece burr piece is 12 minus the number of  $1*1*1$  cubes which have been removed.

**Lemma 9.5.1:** The solid six-piece burr piece has a weight of 12.

*Proof:* The solid piece does not have any cubes removed, hence the weight is  $12-0 = 12$ .

**Lemma 9.5.2:** The lightest usable piece has a weight of 2.

*Proof:* The following image shows the piece with all but 2 cubes removed. Removing any more cubes would render the piece disjoint and thus unusable.

**Lemma 9.5.3:** The weight of a piece is equal to the number of the 12 center cubes which have not been removed.

*Proof:* This follows immediately from the definition.

We have defined the weight of a regular burr as the number of  $1*1$  cubes of the core which are occupied by pieces. Given the above definitions on piece weights, we can now formulate the relationship between piece weights and burr weights:

## **Theorem 10.0:**

The weight of a regular six-piece burr is equal to the sum of the weights of the pieces.

$$WB = WP1 + WP2 + WP3 + WP4 + WP5 + WP6$$

*Proof:* If we would take six pieces of weight 0 (which are disjoint), the core of the burr would be completely empty. The weight of the six-piece burr would then be 0. Now we add one cube to one of the pieces, this will increase the sum of the piece weights by one as well as the burr weight by one. We can continue to do so until we have inserted 32 cubes and the six-piece burr is solid.

**Lemma 10.1:** If the sum of the weight of six given pieces is greater than 32, they cannot be assembled into a six-piece burr.

*Proof:* According to the above Theorem 10.0 the weight of the six-piece burr would be greater than 32, which cannot be according to Lemma 4.1.2.

## Burr Level

We have defined a level as the number of moves required to release a piece. This definition, however, leaves a couple of open questions. First, what constitutes a move, and second, is the move taking the piece out counted or not. If we assume that a move is considered as moving a piece in a particular direction, and we count the removing of the piece as a move as well, then we arrive at the conclusion that the levels published for the Fearsome Four in Puzzles Old & New [Slocum86] are correct. However the level 9 for Peter Marineau's is not. It would be level 10. We have to further add the constraint that, if two independent pieces get moved in the same direction then we consider this only one move. Note that there is a difference between being able to move two pieces independently in the same direction versus two pieces which can only be moved as a unit. Since our computation algorithms favored the behavior to move pieces independently if possible, special code had to be added to the calculation routines to compact the moves if possible. With this further constraint we arrive at the following more exact definition of a level.

### Def 11.0: Level

The level of a regular burr is the number of moves needed to remove the first piece or pieces. A move is counted as one irregardless of how far a piece is moved in one direction. Moving a piece in one direction and then immediately in another is considered two moves. The move to remove the piece is counted for the level also. Moving two or three pieces independently in the same direction immediately after each other is counted as one move. This process is continued for the second, third, etc. piece. The level is then expressed as a.b.c.d.e.f where a is the number of moves to remove piece one, b the number of moves to release piece two and so on.

This definition will however reclassify Philippe Dubois' burr from a level 7.4 to a level 6.4. But we decided to stay with this definition since it gets more of the published burr levels right.

### Def 11.1: Level Type

The level of a regular burr generally changes with the length of the pieces used. Cutler introduced the level type to denote the variation in levels with the length of the pieces. It is a four digit number in hexadecimal notation to denote the four levels at the piece length 6, 8, 10, and 12. For example a type of 9900 indicates that the burr puzzle is of level 9 with pieces of length 6 and 8. It cannot be solved with pieces of length 10 or 12.

*Note:* the java program used to calculate burrs at this site only take into consideration a given piece length at a time. There is no code to calculate all four length in succession. Hence it does not calculate the level type.



## Cutler's Analysis of Solid Six-Piece Burrs

William Cutler was the first to completely analyze solid six-piece burrs in 1978 [Cutler78]. He showed that there are exactly 25 of the possible 59 notchable pieces which can be used to build solid six-piece burrs. With these 25 pieces, a total of 314 solvable burrs can be built. 158 of them use a solid piece as the key.

Cutler further published that there are 369 general pieces which can be used for solid burrs. With these, a total of 119'979 solvable solid burrs are possible. The following table summarizes his findings.

Type	Notchable	All (including Notchable)
<b>1.5</b>	158	6'402
<b>2.4</b>	49	22'840
<b>2.1.3</b>	62	17'574
<b>2.1.1.2</b>	20	525
<b>3.3</b>	16	72'485
<b>2.4-3.3</b>	2	26
<b>2.1.3-3.3</b>	4	28
<b>2.1.1.2-3.3</b>	1	1
<b>3.3-3.3</b>	2	98
<i>Totals</i>	314	119'979

## Cutler's Analysis of Holey Six-Piece Burrs

From the late 1980s to the mid 1990 Bill Cutler and others undertook a complete analysis of all six-piece burrs. Original estimates by Cutler assumed it would take about 62.5 years to run the analysis on a PC AT. With help from others, total time was cut down to 2.5 years. From this analysis we now know that there are roughly 35.65 billion ways to assemble burr puzzles pieces (71.3 billion if mirror images are counted also). Of these 35.65 billion logical assemblies 5.95 billion can be taken apart. The highest level found was level 12. Unfortunately it is not unique, i.e . it has more than one assembly, most of which are of a much lower level. The highest level unique is 10 if the pieces are 8 units long and level 9 if the pieces are 6 units in length. If all pieces are notchable, the highest level is 5 for a unique burr.

Holes	Notchable		General					
	Level 5		Level 8		Level 9		Level 10	
	Count	Unique	Count	Unique	Count	Unique	Count	Unique
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	10	4	0	0	0	0	0	0
4	94	11	15	13	0	0	0	0
5	536	21	1031	828	23	21	0	0
6	1949	40	3730	2128	277	136	0	0
7	4990	31	6754	1681	924	154	13	8
8	9230	22	10353	485	1526	32	36	10
9	12917	10	14415	95	1714	11	58	0
10	14220	0	14264	21	1397	1	96	0
11	12846	0	7816	3	723	0	110	0
12	9431	0	2121	0	192	0	51	0
13	5196	0	268	0	16	0	8	0
14	2014	0	16	0	0	0	1	0
15	547	0	1	0	0	0	0	0
16	95	0	0	0	0	0	0	0
17	9	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0
Total	74085	139	60784	5254	6792	355	373	18