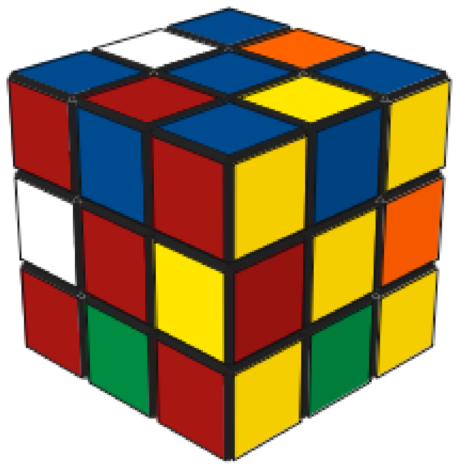
God's Number is 20



R-U2BL-FU-BDFUD-LD2F-RB-DF-U-B-UD-

Superflip, the first position proven to require 20 moves.

Every position of Rubik's Cube™ can be solved in twenty moves or less.

With about 35 CPU-years of idle computer time donated by Google, a team of researchers has essentially solved every position of the Rubik's Cube™, and shown that no position requires more than twenty moves.

Every solver of the Cube uses an algorithm, which is a sequence of steps for solving the Cube. One algorithm might use a sequence of moves to solve the top face, then another sequence of moves to position the middle edges, and so on. There are many different algorithms, varying in complexity and number of moves required, but those that can be memorized by a mortal typically require more than forty moves.

One may suppose God would use a much more efficient algorithm, one that always uses the shortest sequence of moves; this is known as God's Algorithm. The number of moves this algorithm would take in the worst case is called God's Number. At long last, God's Number has been shown to be 20.

It took fifteen years after the introduction of the Cube to find the first position that provably requires twenty moves to solve; it is appropriate that fifteen years after that, we prove that twenty moves suffice for all positions.

A History of God's Number

By 1980, a lower bound of 18 had been established for God's Number by analyzing the number of effectively distinct move sequences of 17 or fewer moves, and finding that there were fewer such sequences than Cube positions. The first upper bound was probably around 80 or so from the algorithm in one of the early solution booklets. This table summarizes the subsequent results.

Date	Lower bound	Upper bound	Gap	Notes and Links
July, 1981	18	52	34	Morwen Thistlethwaite proves 52 moves suffice.
April, 1992	18	42	24	Hans Kloosterman improves this to 42 moves.
May, 1992	18	39	21	Michael Reid shows 39 moves is always sufficient.
May, 1992	18	37	19	Dik Winter lowers this to 37 moves just one day later!

January, 1995	18	29	11	Michael Reid cuts the upper bound to 29 moves by analyzing Kociemba's two-phase algorithm.
January, 1995	20	29	9	Michael Reid proves that the "superflip" position (corners correct, edges placed but flipped) requires 20 moves.
December, 2005	20	28	8	Silviu Radu shows that 28 moves is always enough.
April, 2006	20	27	7	Silviu Radu improves his bound to 27 moves.
May, 2007	20	26	6	Dan Kunkle and Gene Cooperman prove 26 moves suffice.
March, 2008	20	25	5	Tomas Rokicki cuts the upper bound to 25 moves.
April, 2008	20	23	3	Tomas Rokicki and John Welborn reduce it to only 23 moves.
August, 2008	20	22	2	Tomas Rokicki and John Welborn continue down to 22 moves.
July, 2010	20	20	0	Morley Davidson, John Dethridge, Herbert Kociemba, and Tomas Rokicki prove that God's Number for the Cube is exactly 20.

How We Did It

How did we solve all 43,252,003,274,489,856,000 positions of the Cube?

- We partitioned the positions into 2,217,093,120 sets of 19,508,428,800 positions each.
- We reduced the count of sets we needed to solve to 55,882,296 using symmetry and set covering.
- We did not find optimal solutions to each position, but instead only solutions of length 20 or less.
- We wrote a program that solved a single set in about 20 seconds.
- We used about 35 CPU years to find solutions to all of the positions in each of the 55,882,296 sets.

Partitioning

We broke the problem down into 2,217,093,120 smaller problems, each comprising 19,508,428,800 different positions. Each of these subproblems was small enough to fit in the memory of a modern PC, and the way we broke it down (mathematically, using cosets of the group generated by {U,F2,R2,D,B2,L2}, or more concisely, cosets of H) allowed us to solve each set rapidly.

Symmetry

If you take a scrambled Cube and turn it upside down, you have not made it any more difficult; it will still take the same number of moves to solve. Instead of solving both of these positions, you can simply solve one, and then turn the solution upside down for the other. There are 24 different ways you can orient the Cube in space, and another factor of two using a mirror, for a total reduction of a factor of about 48 in the number of positions that need solving. Using similar symmetry arguments and by finding a solution to a large "set cover" problem, we were able to reduce the number of sets that needed solving from 2,217,093,120 down to 55,882,296.

Good vs. Optimal Solutions

	Random positions	Cosets of H
Optimally	0.36	2,000,000
20 moves or less	3,900	1,000,000,000

Solution rate, in positions/second

An optimal solution to a position is one that requires no more moves than is required. Since a position that required 20 moves was already known, we did not need to optimally solve every position; we just needed to find a solution of 20 moves or less for each sequence. This is substantially easier; the table at left show the rate a good desktop PC has when solving random positions.

Fast Coset Solving Program

Using a combination of mathematical tricks and careful programming, we were able to solve a complete coset of H, either optimally, or with sequences of twenty moves or less, on a single desktop PC, at the rates shown in the table at left.

Lots of Computers

Finally, we were able to distribute the 55,882,296 cosets of H among a large number of computers at Google and complete the computation in just a few weeks. Google does not release information on their computer systems, but it would take a good desktop PC (Intel Nehalem, four-core, 2.8GHz) 1.1 billion seconds, or about 35 CPU years, to perform this calculation.

What are the Hardest Positions?

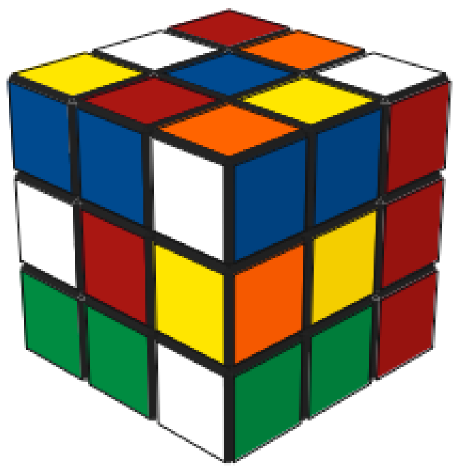
We have known for fifteen years that there are positions that require 20 moves; we have just proved that there are none that require more.

Distance-20 positions are both rare and plentiful; they are rarer than one in a billion positions, yet there are probably more than one hundred million such positions. We do not yet know exactly how many there are. The table on the right gives the count of positions at each distance; for distances 16 and greater, the number given is just an estimate. Our research

Distance	Count of Positions
0	1
1	18
2	243
3	3,240
4	43,239
5	574,908

has confirmed the prior results for entries 0 through 14 below, and the entry for 15 is a new result. We hope to have that independently confirmed by another researcher within the month.

To date we have found about twelve million distance-20 positions. The following position was the hardest for our programs to solve:



FU-F2D-BUR-F-LD-R-U-LUB-D2R-FU2D2

The hardest position for our programs.

6	7,618,438
7	100,803,036
8	1,332,343,288
9	17,596,479,795
10	232,248,063,316
11	3,063,288,809,012
12	40,374,425,656,248
13	531,653,418,284,628
14	6,989,320,578,825,358
15	91,365,146,187,124,313
16	about 1,100,000,000,000,000,000
17	about 12,000,000,000,000,000,000
18	about 29,000,000,000,000,000,000
19	about 1,500,000,000,000,000,000
20	about 300,000,000

Contact

Our group consists of Morley Davidson, a mathematician from Kent State University, John Dethridge, an engineer at Google in Mountain View, Herbert Kociemba, math teacher from Darmstadt, Germany, and Tomas Rokicki, a programmer from Palo Alto, California. Email may be sent to rokicki@gmail.com or to davidson@math.kent.edu.

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Thanks to Werner Randelshofer for use of the Cube applet on this page.