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Square 1 Puzzle

-- Fred Curtis

[Puzzle Geometry] [What is an arrangement?] [How many arrangements are there?]
[Grouping similar arrangements] [Canonical Shapes] [Program Sources] [Chronology]

"Square 1" is a combinatorial puzzle (similar to "Rubik's Cube") which appeared around 1992. You can see some pictures of Square 1 on the following pages:

- Square 1 by Chris and Kori - includes solution devised by Timo Jokitalo
- Directions for solving the Square 1 - includes puzzle solution devised by Andrew Arensburger and Christian Eggermont

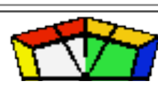
You can play with the Virtual Square 1 Puzzle on this site.

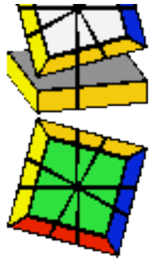
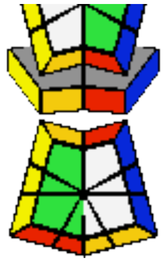
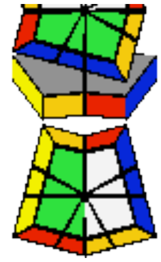
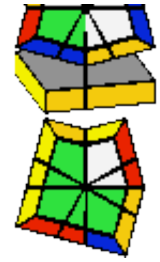
This page is a collection of notes on writing a computer program to solve the Square 1 puzzle.

Puzzle Geometry

Despite admitting some very strange-looking shapes, the Square 1 is essentially two outer layers -- each like a pie made of 60° and 30° segments -- separated by a middle layer of 2 large pieces. The outer layers can be rotated around the "pie" axis, and when the outer layers are lined up correctly the middle layer can be flipped, a move which transfers some pieces between the outer layers. For example:

Table 1 - Some Square 1 moves

3D View:							
		Flip ~		Rotate top 90° (as viewed		Flip	

Symbolic View:		->		from above) ->			

What is an arrangement?

How many ways are there to arrange the pieces of the Square 1 puzzle? That depends on when two physical arrangements of pieces in the puzzle are regarded as the same. Some observations that lump similar arrangements together are:

- Picking up the puzzle and turning it upside down or on its side doesn't change the positions of the pieces relative to one another, and doesn't get any closer to or further from a solution. To group all of these 'identical' arrangements together, regard some piece as fixed in space, with all the other pieces arranged relative to the fixed piece. All versions of the puzzle I've seen have 'Square 1' written on one of the large middle-layer pieces -- regard this piece as fixed in space.
- Consider only arrangements where the two middle-layer pieces are lined up in the same plane, that is, arrangements where the puzzle is not partway through a 'flip' move. Using the fixed middle-layer piece described above, we can define the "top" layer as the outer layer closest to the top of the writing on the fixed piece. Call the other outer layer the "bottom" layer.
- The puzzle is jumbled (and solved) by repeatedly rotating the top and bottom layers and making 'flip' moves. If we have some particular arrangement of pieces then just rotating the top or bottom layers doesn't make the puzzle any easier or harder to solve. To this extent we can ignore the exact positions of the pieces in the top and bottom layers and just consider which pieces are in each layer and what order they are in.

With these simplifications, an arrangement around a fixed middle-layer piece can be described as:

- A list of pieces in the top layer listed clockwise viewed from above
- A list of pieces in the bottom layer listed clockwise viewed from the below
- The state of the middle layer (square or not square)

How many arrangements are there?

How many of these arrangements are there? The most straightforward way to count them is to consider a "blank" version of the puzzle where all the labels/colours have been removed from the 60° and 30° pieces, list all the possible arrangements using these "blank" pieces, then for each "blank" arrangement count the number of ways it can be "coloured" by putting the labels/colours back.

Table 2 below shows all the ways that the blank pieces can be arranged in the top or bottom layer.

Table 2 - Arrangements of 60° and 30° pieces in a top/bottom layer
[Perl source code]

60° pieces 'A' in layer	30° pieces 'a' in layer	ID	Layer shape	Reflection Symmetry	Rotation Symmetries	Permutation Multiplier	Permutation Sum
6	0	6-1	AAAAAA	Y	6	1/6	1/6
5	2	5-1	AAAAAaa	Y	1	1	3
		5-2	AAAAaAa	Y	1	1	
		5-3	AAAaAAa	Y	1	1	
4	4	4-1	AAAAaaaa	Y	1	1	35/4 = 8 3/4
		4-2	AAAaAaaa	N	1	1	
		4-3	AAAaaAaa	Y	1	1	
		4-4	AAAaaaAa	N	1	1	
		4-5	AAaAAaaa	Y	1	1	
		4-6	AAaAaAaa	N	1	1	
		4-7	AAaAaaAa	Y	1	1	
		4-8	AAaaAAaa	Y	2	1/2	
		4-9	AAaaAaAa	N	1	1	
		4-10	AaAaAaAa	Y	4	1/4	
3	6	3-1	AAAaaaaaa	Y	1	1	28/3 = 9 1/3
		3-2	AAaAaaaaa	N	1	1	
		3-3	AAaaAaaaa	N	1	1	
		3-4	AAaaaAaaa	Y	1	1	
		3-5	AAaaaaAaa	N	1	1	
		3-6	AAaaaaaAa	N	1	1	
		3-7	AaAaAaaaa	Y	1	1	
		3-8	AaAaaAaaa	N	1	1	
		3-9	AaAaaaAaa	N	1	1	
		3-10	AaaAaaAaa	Y	3	1/3	
2	8	2-1	AAaaaaaaaa	Y	1	1	9/2 = 4 1/2
		2-2	AaAaaaaaaa	Y	1	1	
		2-3	AaaAaaaaaa	Y	1	1	
		2-4	AaaaAaaaaa	Y	1	1	
		2-5	AaaaaAaaaa	Y	2	1/2	

Table 3 - Permutation counts for top & bottom layers, factoring out rotations

No. of 60° pieces in top layer	No. of 60° pieces in bottom layer	Product of permutation multipliers
6	2	1/6 * 9/2 = 3/4
5	3	3 * 28/3 = 28
4	4	35/4 * 35/4 = 1225/16 = 76 9/16
3	5	3 * 28/3 = 28
2	6	1/6 * 9/2 = 3/4
	Total:	2145/16 = 134 1/16

Each of the blank arrangements formed using the layers from Table 2 can be coloured in 8!·8! ways, but care has to be taken to avoid duplicates. For example, a blank shape with a layer 6-1 on top and layer 2-1 on the bottom can be coloured 8!·8! ways, but this must be divided by 6 to take into account the rotation symmetry of the top layer.

Each of the 2145/16 shapes (factoring out rotations - see Table 3) can be 'coloured' in 8!·8! = 1625702400 ways, and the middle pieces may or may not form a square (another factor of 2), so there is a total of 435891456000 arrangements (about 2^39). A web search on this number yielded one other calculation of the number of Square 1 states (by Michael C. Masonjones, dated 28 May 1996) which agrees!

Grouping similar arrangements

In writing a computer program to solve (unscramble) the puzzle, the more arrangements that can be lumped together as similar (canonical) arrangements, the better -- assuming that the time & storage space saved by dealing with fewer arrangements isn't overwhelmed by the time taken to decide whether two arrangements are the same.

To understand some ways of drastically reducing the number of arrangements to consider, it is convenient to have labels for the 60° and 30° pieces in the puzzle, and a description of what constitutes a solution for an arrangement:

- Imagine a Square 1 puzzle in the solved state.
- Label the pieces on the top layer, viewed clockwise from the top, with the labels $T_0,t_0,T_1,t_1,T_2,t_2,T_3,t_3$, where the T_i are 60° pieces and the t_i are 30° pieces.
- Label the pieces on the bottom layer, viewed clockwise from the bottom, with the labels $B_0,b_0,B_1,b_1,B_2,b_2,B_3,b_3$, where the B_i are 60° pieces and the b_i are 30° pieces.
- A solution for an arrangement A is a sequence of steps, each of which looks like:
    - Rotate the top and bottom layers until [some list of pieces currently in the layers] are above/below the fixed middle-layer piece.

with a flip move to be made between each step.

Some ways of grouping arrangements for the purposes of solving the puzzle are:

- **Sym1** - *Group arrangements together which correspond to rotations of the top and/or bottom layer of the solved puzzle.*

  Suppose we have an arrangement A and a solution S that returns A to the solved state.

  Make a copy of A, but permute the pieces $T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_0$ and $t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_0$ and call the resulting copy A_2. Make a copy of S, apply the same label permutation and call the resulting solution S_2. Apply S_2 to A_2 and the result will be a solved puzzle with the top rotated by 90°.

  That is (apart from a final rotation of the top layer), the same solution pattern suffices to solve both A and A_2. Similarly if the permutation is applied again, and if a similar permutation is applied to the B_i/b_i pieces.

- **Sym2** - *Group arrangements together which correspond to swapping top & bottom layers and exchanging T_i<=>B_i and t_i<=>b_i.*

  Again, consider an arrangement A and a solution S which take A to the solved state. Make a copy of A but exchange each T_i<=>B_i and each t_i<=>b_i and call the copy A_2. Apply the same permutation to S and call the result S_2. Apply S_2 to A_2 and the result will be an arrangement with all the top pieces in the solved order (but location in the bottom layer) and all the bottom pieces in the solved order (but location in the top layer), or equivalently (by flipping the result upside down), the top and bottom layers in their correctly solved state but the middle layer of the puzzle inverted.

  Make a copy of A_2 but with the middle layer inverted and call this copy A_3. Apply S_2 to A_3 and the result will be a solved puzzle (but inverted).

  The same solution pattern S suffices to solve both A and A_3. A_3 is obtained from A by swapping the top and bottom layers and exchanging T_i<=>B_i and t_i<=>b_i (and then inverting the result, which doesn't change the relative positions of the pieces).

  This grouping implies that if we consider arrangements with different Table 2 layer patterns (e.g. top layer 6-1 and bottom layer 2-2) then we can ignore arrangements with the layer patterns exchanged (e.g. top layer 2-2 and bottom layer 6-1).

  For arrangements with the same layer patterns (i.e. top and bottom layers both 4-something) the grouping gathers arrangements such as [solved apart from swapping two top corners] and [solved apart from swapping two corresponding bottom corners].

- **Sym3** - *Group arrangements together which correspond to mirror reflections.*

  Again, consider an arrangement A and a solution S which take A to the solved state. Make a copy of A reflected in a (vertical) mirror and call the copy A_2. Apply S to A_2. The result will be a puzzle with all the t_i on the top layer (but ordered $T_0,t_3,T_3,t_2,T_2,t_1,T_1,t_0$ viewed clockwise from the top) and all the b_i on the bottom layer (but ordered $B_0,b_3,B_3,b_2,B_2,b_1,B_1,b_0$ viewed clockwise from the bottom).

  Make a copy of A_2 but swap t_0 <=> t_3, T_1 <=> T_3, t_1 <=> t_2, b_0 <=> b_3, B_1 <=> B_3, b_1 <=> b_2. Call this copy A_3. Make a copy of S applying the same permutation and call the result S_2. Apply S_2 to A_3 and the result will be a solved puzzle.

  The same solution pattern suffices to solve both A and A_3. A_3 is obtained from A by forming an image in a (vertical) mirror and swapping t_0 <=> t_3, T_1 <=> T_3, t_1 <=> t_2, b_0 <=> b_3, B_1 <=> B_3, b_1 <=> b_2.

  This grouping implies that if we consider arrangements where at least one of the top or bottom layer pattern (see Table 2) lacks reflection symmetry (e.g., pattern 4-2 on top and 4-10 on bottom) then we can ignore arrangements with the mirror image patterns (e.g., pattern 4-4 on top and 4-10 on bottom).

  For arrangements where both top and bottom layer patterns have reflection symmetry, the grouping gathers arrangements such as [solved apart from switching three top pieces clockwise] and [solved apart from switching corresponding pieces anticlockwise].

- **Sym4** - There is a combination of Sym2 and Sym3 which applies where the top layer pattern is the reflection of the bottom layer pattern (e.g. top pattern 4-2, bottom pattern 4-4).

Canonical Shapes

With the symmetries described above, the number of shapes considered can be reduced to the 65 listed in Table 4, ignoring the state of the middle layer. Each of these shapes can be labelled in 8!·8! ways by replacing each 'A' with a T_i or B_i, and each 'a' with a t_i or b_i. This number can be reduced applying the Sym1 rotation of pieces and assuming the first occurrence of a T_i is T_0, and the first occurrence of B_i is B_0.

Table 4 - Canonical Shapes
(Ignoring state of middle-layer)

Shape ID	Top Layer	Top Rotsym	Bottom Layer	Bottom Rotsym	Sym2	Sym3	Sym4	Symmetry Group Size	Unique TtBb Templates
1	AAAAAA	6	AAaaaaaaaa	1	N	Y	N	32	?
2	AAAAAA	6	AaAaaaaaaa	1	N	Y	N	32	?
3	AAAAAA	6	AaaAaaaaaa	1	N	Y	N	32	?
4	AAAAAA	6	AaaaAaaaaa	1	N	Y	N	32	?
5	AAAAAA	6	AaaaaAaaaa	2	N	Y	N	32	?
6	AAAAAaa	1	AAAaaaaaa	1	N	Y	N	32	?
7	AAAAAaa	1	AAaAaaaaa	1	N	N	N	16	?
8	AAAAAaa	1	AAaaAaaaa	1	N	N	N	16	?
9	AAAAAaa	1	AAaaaAaaa	1	N	Y	N	32	?
10	AAAAAaa	1	AaAaAaaaa	1	N	Y	N	32	?
11	AAAAAaa	1	AaAaaAaaa	1	N	N	N	16	?
12	AAAAAaa	1	AaaAaaAaa	3	N	Y	N	32	?
13	AAAAaAa	1	AAAaaaaaa	1	N	Y	N	32	?
14	AAAAaAa	1	AAaAaaaaa	1	N	N	N	16	?
15	AAAAaAa	1	AAaaAaaaa	1	N	N	N	16	?
16	AAAAaAa	1	AAaaaAaaa	1	N	Y	N	32	?
17	AAAAaAa	1	AaAaAaaaa	1	N	Y	N	32	?
18	AAAAaAa	1	AaAaaAaaa	1	N	N	N	16	?
19	AAAAaAa	1	AaaAaaAaa	3	N	Y	N	32	?
20	AAAaAAa	1	AAAaaaaaa	1	N	Y	N	32	?
21	AAAaAAa	1	AAaAaaaaa	1	N	N	N	16	?
22	AAAaAAa	1	AAaaAaaaa	1	N	N	N	16	?
23	AAAaAAa	1	AAaaaAaaa	1	N	Y	N	32	?
24	AAAaAAa	1	AaAaAaaaa	1	N	Y	N	32	?
25	AAAaAAa	1	AaAaaAaaa	1	N	N	N	16	?
26	AAAaAAa	1	AaaAaaAaa	3	N	Y	N	32	?
27	AAAAaaaa	1	AAAAaaaa	1	Y	Y	Y	64	?
28	AAAAaaaa	1	AAAaAaaa	1	N	N	N	16	?
29	AAAAaaaa	1	AAAaaAaa	1	N	Y	N	32	?
30	AAAAaaaa	1	AAaAAaaa	1	N	Y	N	32	?
31	AAAAaaaa	1	AAaAaAaa	1	N	N	N	16	?
32	AAAAaaaa	1	AAaAaaAa	1	N	Y	N	32	?

33	AAAAaaaa	1	AAaaAAaa	2	N	Y	N	32	?
34	AAAAaaaa	1	AaAaAaAa	4	N	Y	N	32	?
35	AAAaAaaa	1	AAAaAaaa	1	Y	N	N	32	?
36	AAAaAaaa	1	AAAaaAaa	1	N	N	N	16	?
37	AAAaAaaa	1	AAAaaaAa	1	N	N	Y	32	?
38	AAAaAaaa	1	AAaAAaaa	1	N	N	N	16	?
39	AAAaAaaa	1	AAaAaAaa	1	N	N	N	16	?
40	AAAaAaaa	1	AAaAaaAa	1	N	N	N	16	?
41	AAAaAaaa	1	AAaaAAaa	2	N	N	N	16	?
42	AAAaAaaa	1	AAaaAaAa	1	N	N	N	16	?
43	AAAaAaaa	1	AaAaAaAa	4	N	N	N	16	?
44	AAAaaAaa	1	AAAaaAaa	1	Y	Y	Y	64	?
45	AAAaaAaa	1	AAaAAaaa	1	N	Y	N	32	?
46	AAAaaAaa	1	AAaAaAaa	1	N	N	N	16	?
47	AAAaaAaa	1	AAaAaaAa	1	N	Y	N	32	?
48	AAAaaAaa	1	AAaaAAaa	2	N	Y	N	32	?
49	AAAaaAaa	1	AaAaAaAa	4	N	Y	N	32	?
50	AAaAAaaa	1	AAaAAaaa	1	Y	Y	Y	64	?
51	AAaAAaaa	1	AAaAaAaa	1	N	N	N	16	?
52	AAaAAaaa	1	AAaAaaAa	1	N	Y	N	32	?
53	AAaAAaaa	1	AAaaAAaa	2	N	Y	N	32	?
54	AAaAAaaa	1	AaAaAaAa	4	N	Y	N	32	?
55	AAaAaAaa	1	AAaAaAaa	1	Y	N	N	32	?
56	AAaAaAaa	1	AAaAaaAa	1	N	N	N	16	?
57	AAaAaAaa	1	AAaaAAaa	2	N	N	N	16	?
58	AAaAaAaa	1	AAaaAaAa	1	N	N	Y	32	?
59	AAaAaAaa	1	AaAaAaAa	4	N	N	N	16	?
60	AAaAaaAa	1	AAaAaaAa	1	Y	Y	Y	64	?
61	AAaAaaAa	1	AAaaAAaa	2	N	Y	N	32	?
62	AAaAaaAa	1	AaAaAaAa	4	N	Y	N	32	?
63	AAaaAAaa	2	AAaaAAaa	2	Y	Y	Y	64	?
64	AAaaAAaa	2	AaAaAaAa	4	N	Y	N	32	?
65	AaAaAaAa	4	AaAaAaAa	4	Y	Y	Y	64	?
Total of reciprocals of symmetry group sizes:								2.65625	

Program Sources

Selected CGI script sources for the Virtual Square 1 Puzzle on this site:

- f2cgi.pm -- Generic CGI routines for grabbing parameters and writing HTTP headers
- xform3d.pm -- Routines for performing 3D transforms (rotations, translations, etc) on 3D points
- poly3d.pm -- Routines for manipulating lists of 3D points representing coloured polygons
- obj3d.pm -- Routines for manipulating lists of 3D coloured polygons representing objects
- gd3d.pm -- Routines for drawing 3D objects using the perl GD module
- sq1.pm -- Routines for manipulating logical/combinatorial Square 1 puzzle
- sq13d.pm -- Routines for drawing a Square 1 puzzle using GD library

Miscellaneous source:

- Perl source code for Tables 2 & 4

Chronology

- Feb-March 2002 - Re-encountered Square 1 puzzle at Geoff's place. Reproduced by Perl scripts the results of 90 distinct shapes (65 by folding mirror images) + max diameter 7 of state graph.
- Circa 18 March 2002 - Back-of-the-envelope calcs + some symmetry counts in Perl suggest an encoding of canonical state in just under 32 bits.
- 21 March 2002 - Started this web page
- 26 March 2002 - Began writing a small 3D polygon layer over the Perl GD library to render Square 1 states. Nothing done over Easter break.
- 10 April 2002 - Gave up trying to z-order the polygons for rendering & wrote a pure perl 3D polygon scan-line renderer (see gd3d.pm) to avoid using rendering libraries that I couldn't install on the web server
- 18 April 2002 - 3D images rendering acceptably ; trying to coming up with a usable 2D/plan view
- 23 April 2002 - CGI scripts now allow change of colour / size / view rotations
- 7 May 2002 - Finally finished Virtual Square-1 CGI, but still want to implement image cache
- 11 May 2002 - Realised there was a gaping hole in my scheme to represent canonical states using only 32 bits
- 14 May 2002 - Laid up at home with a sick foot. Began writing section on puzzle geometry. Began mods to CGI script to return puzzle to cube shape with minimal flips.
- 15 July 2002 - Resumed writing page after 2 months of other distractions. Wrote Table 2 & Table 3 and calculations of number of possible configurations.
- 17 July 2002 - Finished writing about ways of grouping arrangements to cut solution computation time. Realised a nice way to specify a flip move was as the two pieces delimiting the segments which stay over the fixed middle-layer piece -- this is invariant under mirror- image reflections and top/bottom layer exchanges.
- Began rewording using 'shape' to denote an arrangement of of 'blank' pieces, and 'arrangement' to denote an arrangement of 'labelled' pieces.
- 24 July 2002 - Started section on canonical shapes. Wrote perl code for Table 4. Came up with nice representation for arrangement transformations.

- 28 July 2002 - Wrote code to generate symmetry group for each canonical shape.
- 30 Dec 2004 - Started working on this problem again after 2.5 years of being distracted with other matters :-). Began documenting section on transformations.
- 23 Jan 2004 - Reworded some introductory remarks, characterised a solution as steps of rotating certain pieces above/below the fixed piece with flip moves between each step.
