Search Site Go

Algebra
Applied Mathematics
Calculus and Analysis
Discrete Mathematics
Foundations of Mathematics
Geometry
History and Terminology
Number Theory
Probability and Statistics
Recreational Mathematics
Topology

Alphabetical Index
Interactive Entries
Random Entry
New in *MathWorld*

MathWorld Classroom

About *MathWorld*
Send a Message to the Team

Order book from Amazon

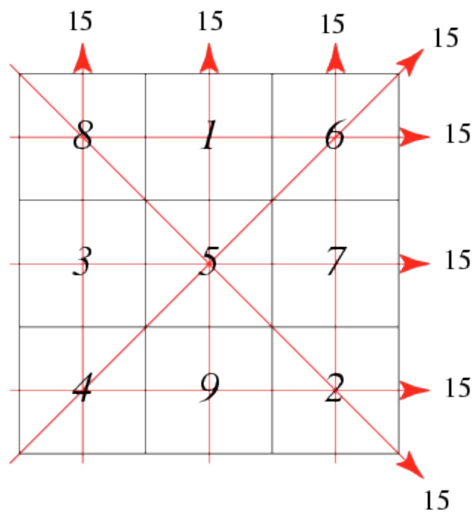
Last updated:
12,754 entries
Fri Jan 25 2008

Created, developed, and nurtured by Eric Weisstein at Wolfram Research

Recreational Mathematics > Magic Figures > Magic Squares
Foundations of Mathematics > Mathematical Problems > Unsolved Problems
Algebra > Linear Algebra > Matrices > Matrix Types

Magic Square

COMMENT On this Page
EXPLORE THIS TOPIC IN The *MathWorld* Classroom
DOWNLOAD *Mathematica* Notebook



A magic square is a square array of numbers consisting of the distinct positive integers $1, 2, ..., n^2$ arranged such that the sum of the n numbers in any horizontal, vertical, or *main* diagonal line is always the same number (Kraitchik 1952, p. 142; Andrews 1960, p. 1; Gardner 1961, p. 130; Madachy 1979, p. 84; Benson and Jacobi 1981, p. 3; Ball and Coxeter 1987, p. 193), known as the magic constant

$$M_2(n) = \frac{1}{n} \sum_{k=1}^{n^2} k = \frac{1}{2} n (n^2 + 1).$$

If every number in a magic square is subtracted from $n^2 + 1$, another magic square is obtained called the complementary magic square. A square consisting of consecutive numbers starting with 1 is sometimes known as a "normal" magic square.

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

The unique normal square of order three was known to the ancient Chinese, who called it the Lo Shu. A version of the order-4 magic square with the numbers 15 and 14 in adjacent middle columns in the bottom row is called Dürer's magic square. Magic squares of order 3 through 8 are shown above.

The magic constant for an nth order general magic square starting with an integer A and with entries in an increasing arithmetic series with difference D between terms is

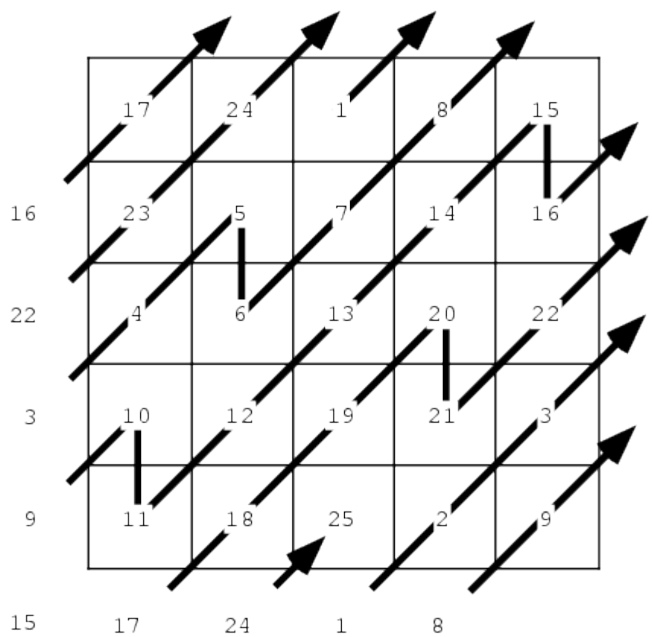
$$M_2(n; A, D) = \frac{1}{2} n [2a + D(n^2 - 1)]$$

(Hunter and Madachy 1975).

It is an unsolved problem to determine the number of magic squares of an arbitrary order, but the number of distinct magic squares (excluding those obtained by rotation and reflection) of order $n = 1$, 2, ... are 1, 0, 1, 880, 275305224, ... (Sloane's A006052; Madachy 1979, p. 87). The 880 squares of order four were enumerated by Frénicle de Bessy (1693), and are illustrated in Berlekamp *et al.* (1982, pp. 778-783). The number of 5×5 magic squares was computed by R. Schroeppel in 1973. The number of 6×6 squares is not known, but Pinn and Wieczerkowski (1998) estimated it to be $(1.7745 \pm 0.0016) \times 10^{19}$ using Monte Carlo simulation and methods from statistical mechanics. Methods for enumerating magic squares are discussed by Berlekamp *et al.* (1982) and on the MathPages website.

A square that fails to be magic only because one or both of the main diagonal sums do not equal the magic constant is called a semimagic square. If *all* diagonals (including those obtained by wrapping around) of a magic square sum to the magic constant, the square is said to be a panmagic square (also called a diabolic square or pandiagonal square). If replacing each number n_i by its square n_i^2 produces another magic square, the square is said to be a bimagic square (or doubly magic square). If a square is magic for n_i, n_i^2, and n_i^3, it is called a trimagic square (or trebly magic square). If all pairs of numbers symmetrically opposite the center sum to $n^2 + 1$, the square is said to be an associative magic square.

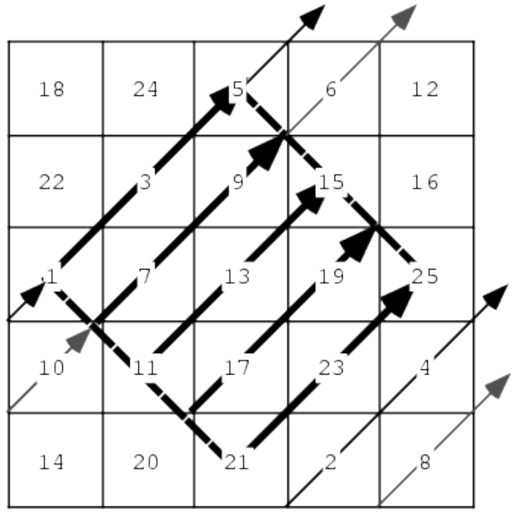
Squares that are magic under multiplication instead of addition can be constructed and are known as multiplication magic squares. In addition, squares that are magic under both addition *and* multiplication can be constructed and are known as addition-multiplication magic squares (Hunter and Madachy 1975).



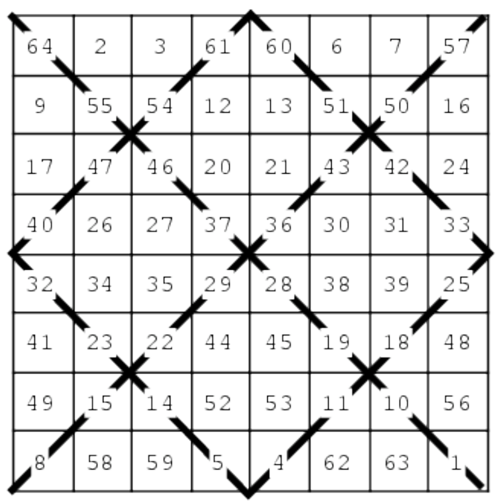
Kraitchik (1942) gives general techniques of constructing even and odd squares of order n. For n odd, a very straightforward technique known as the Siamese method can be used, as illustrated above (Kraitchik 1942, pp. 148-149). It begins by placing a 1 in any location (in the center square of the top row in the above example), then incrementally placing subsequent numbers in the square one unit above and to the right. The counting is wrapped around, so that falling off the top returns on the bottom and falling off the right returns on the left. When a square is encountered that is already filled, the next number is instead placed *below* the previous one and the method continues as before. The method, also called de la Loubere's method, is purported to have been first reported in the West when de la Loubere returned to France after serving as ambassador to Siam.

A generalization of this method uses an "ordinary vector" (x, y) that gives the offset for each noncolliding move and a "break vector" (u, v) that gives the offset to introduce upon a collision. The standard Siamese method therefore has ordinary vector (1, -1) and break vector (0, 1). In order for this to produce a magic square, each break move must end up on an unfilled cell. Special classes of magic squares can be constructed by considering the absolute sums $|u + v|$, $|(u - x) + (v - y)|$, $|u - v|$, and $|(u - x) - (v - y)| = |u + y - x - v|$. Call the set of these numbers the sumdiffs (sums and differences). If all sumdiffs are relatively prime to n and the square is a magic square, then the square is also a panmagic square. This theory originated with de la Hire. The following table gives the sumdiffs for particular choices of ordinary and break vectors.

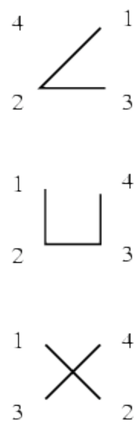
Ordinary Vector	Break Vector	Sumdiffs	Magic Squares	Panmagic Squares
(1, -1)	(0, 1)	(1, 3)	$2k+1$	none
(1, -1)	(0, 2)	(0, 2)	$6k\pm 1$	none
(2, 1)	(1, -2)	(1, 2, 3, 4)	$6k\pm 1$	none
(2, 1)	(1, -1)	(0, 1, 2, 3)	$6k\pm 1$	$6k\pm 1$
(2, 1)	(1, 0)	(0, 1, 2)	$2k+1$	none
(2, 1)	(1, 2)	(0, 1, 2, 3)	$6k\pm 1$	none



A second method for generating magic squares of odd order has been discussed by J. H. Conway under the name of the "lozenge" method. As illustrated above, in this method, the odd numbers are built up along diagonal lines in the shape of a diamond in the central part of the square. The even numbers that were missed are then added sequentially along the continuation of the diagonal obtained by wrapping around the square until the wrapped diagonal reaches its initial point. In the above square, the first diagonal therefore fills in 1, 3, 5, 2, 4, the second diagonal fills in 7, 9, 6, 8, 10, and so on.



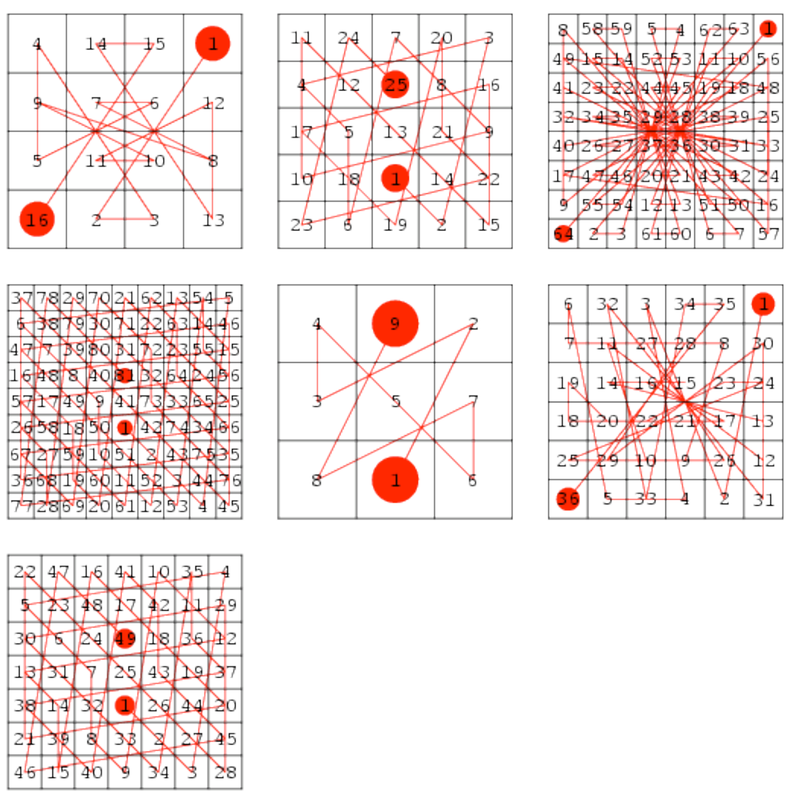
An elegant method for constructing magic squares of doubly even order $n=4m$ is to draw xs through each 4×4 subsquare and fill all squares in sequence. Then replace each entry a_{ij} on a crossed-off diagonal by $(n^2+1)-a_{ij}$ or, equivalently, reverse the order of the crossed-out entries. Thus in the above example for $n=8$, the crossed-out numbers are originally 1, 4, ..., 61, 64, so entry 1 is replaced with 64, 4 with 61, etc.



68	65	96	93	4	1	32	29	60	57
L		L		L		L		L	
66	67	94	95	2	3	30	31	58	59
92	89	20	17	28	25	56	53	64	61
L		L		L		L		L	
90	91	18	19	26	27	54	55	62	63
16	13	24	21	49	52	80	77	88	85
L		L		U		L		L	
14	15	22	23	50	51	78	79	86	87
37	40	45	48	76	73	81	84	9	12
U		U		L		U		U	
38	39	46	47	74	75	82	83	10	11
41	44	69	72	97	100	5	8	33	36
X		X		X		X		X	
43	42	71	70	99	98	7	6	35	34

A very elegant method for constructing magic squares of singly even order $n=4m+2$ with $m \geq 1$ (there is no magic square of order 2) is due to J. H. Conway, who calls it the "LUX" method. Create an array consisting of $m+1$ rows of Ls, 1 row of Us, and $m-1$ rows of Xs, all of length $n/2=2m+1$. Interchange the middle U with the L above it. Now generate the magic square of order $2m+1$ using the Siamese method centered on the array of letters (starting in the center square of the top row), but fill each set of four squares surrounding a letter sequentially according to the order prescribed by the letter. That order is illustrated on the left side of the above figure, and the completed square is illustrated to the right. The "shapes" of the letters L, U, and X naturally suggest the filling order, hence the name of the algorithm.

Variations on magic squares can also be constructed using letters (either in defining the square or as entries in it), such as the alphamagic square and templar magic square.



Various numerological properties have also been associated with magic squares. Pivari associates the squares illustrated above with Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon, respectively. Attractive patterns are obtained by connecting consecutive numbers in each of the squares (with the exception of the Sun magic square).

SEE ALSO: Addition-Multiplication Magic Square Alphamagic Square, Antimagic Square, Associative Magic Square, Bimagic Square, Border Square, Dürer's Magic Square, Euler Square, Franklin Magic Square, Gnomon Magic Square, Heterosquare, Latin Square, Magic Circles, Magic Constant, Magic Cube, Magic Hexagon, Magic Labeling, Magic Series, Magic Tesseract, Magic Tour, Multimagic Square, Multiplication Magic Square, Panmagic Square, Semimagic Square, Talisman Square, Templar Magic Square, Trimagic Square. [Pages Linking Here]

REFERENCES:

Abe, G. "Unsolved Problems on Magic Squares." *Disc. Math.* **127**, 3-13, 1994.

Alejandre, S. "Suzanne Alejandre's Magic Squares." http://mathforum.org/alejandre/magic.square.html.

Andrews, W. S. *Magic Squares and Cubes, 2nd rev. ed.* New York: Dover, 1960.

Andrews, W. S. and Sayles, H. A. "Magic Squares Made with Prime Numbers to have the Lowest Possible Summations." *Monist* **23**, 623-630, 1913.

Ball, W. W. R. and Coxeter, H. S. M. "Magic Squares." Ch. 7 in *Mathematical Recreations and Essays, 13th ed.* New York: Dover, 1987.

Barnard, F. A. P. "Theory of Magic Squares and Cubes." *Memoirs Natl. Acad. Sci.* **4**, 209-270, 1888.

Benson, W. H. and Jacoby, O. *Magic Cubes: New Recreations.* New York: Dover, 1981.

Benson, W. H. and Jacoby, O. *New Recreations with Magic Squares.* New York: Dover, 1976.

Berlekamp, E. R.; Conway, J. H; and Guy, R. K. *Winning Ways for Your Mathematical Plays, Vol. 2: Games in Particular.* London: Academic Press, 1982.

Chabert, J.-L. (Ed.). "Magic Squares." Ch. 2 in *A History of Algorithms: From the Pebble to the Microchip.* New York: Springer-Verlag, pp. 49-81, 1999.

Danielsson, H. "Magic Squares." http://www.magic-squares.de/magic.html.

Flannery, S. and Flannery, D. *In Code: A Mathematical Journey.* London: Profile Books, pp. 16-24, 2000.

Frénicle de Bessy, B. "Des quarrez ou tables magiques. Avec table generale des quarrez magiques de quatre de costé." In *Divers Ouvrages de Mathématique et de Physique, par Messieurs de l'Académie Royale des Sciences* (Ed. P. de la Hire). Paris: De l'imprimerie Royale par Jean Anisson, pp. 423-507, 1693. Reprinted as *Mem. de l'Acad. Roy. des Sciences* **5** (pour 1666-1699), 209-354, 1729.

Fults, J. L. *Magic Squares.* Chicago, IL: Open Court, 1974.

Gardner, M. "Magic Squares." Ch. 12 in *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 130-140, 1961.

Gardner, M. "Magic Squares and Cubes." Ch. 17 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, pp. 213-225, 1988.

Grogono, A. W. "Magic Squares by Grog." http://www.grogono.com/magic/.

Hawley, D. "Magic Squares." http://www.nrich.maths.org.uk/mathsf/journalf/aug98/art1/.

Heinz, H. "Downloads." http://www.geocities.com/~harveyh/downloads.htm.

Heinz, H. "Magic Squares." http://www.geocities.com/CapeCanaveral/Launchpad/4057/magicsquare.htm.

Heinz, H. and Hendricks, J. R. *Magic Square Lexicon: Illustrated.* Self-published, 2001. http://www.geocities.com/~harveyh/BookSale.htm.

Hirayama, A. and Abe, G. *Researches in Magic Squares.* Osaka, Japan: Osaka Kyoikutosho, 1983.

Horner, J. "On the Algebra of Magic Squares, I., II., and III." *Quart. J. Pure Appl. Math.* **11**, 57-65, 123-131, and 213-224, 1871.

Hunter, J. A. H. and Madachy, J. S. "Mystic Arrays." Ch. 3 in *Mathematical Diversions.* New York: Dover, pp. 23-34, 1975.

Kanada, Y. "Magic Square Page." http://www.kanadas.com/puzzles/magic-square.html.

Kraitchik, M. "Magic Squares." Ch. 7 in *Mathematical Recreations.* New York: Norton, pp. 142-192, 1942.

UPDATE THIS LINK Lei, A. "Magic Square, Cube, Hypercube." http://www.cs.ust.hk/~philipl/magic/.

Madachy, J. S. "Magic and Antimagic Squares." Ch. 4 in *Madachy's Mathematical Recreations.* New York: Dover, pp. 85-113, 1979.

MathPages. "Solving Magic Squares." http://www.mathpages.com/home/kmath295.htm.

Moran, J. *The Wonders of Magic Squares.* New York: Vintage, 1982.

Pappas, T. "Magic Squares," "The 'Special' Magic Square," "The Pyramid Method for Making Magic Squares," "Ancient Tibetan Magic Square," "Magic 'Line.'," and "A Chinese Magic Square." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 82-87, 112, 133, 169, and 179, 1989.

Peterson, I. "Ivar Peterson's MathLand: More than Magic Squares." http://www.maa.org/mathland/mathland_10_14.html.

Pickover, C. A. *The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures Across Dimensions.* Princeton, NJ: Princeton University Press, 2002.

Pinn, K. and Wieczerkowski, C. "Number of Magic Squares from Parallel Tempering Monte Carlo." *Int. J. Mod. Phys. C* **9**, 541-547, 1998. http://arxiv.org/abs/cond-mat/9804109/.

Pivari, F. "Nice Examples." http://www.geocities.com/CapeCanaveral/Lab/3469/examples.html.

Pivari, F. "Create Your Magic Square." http://www.pivari.com/squaremaker.html.

Sloane, N. J. A. Sequence A006052/M5482 in "The On-Line Encyclopedia of Integer Sequences."

Suzuki, M. "Magic Squares." http://mathforum.org/te/exchange/hosted/suzuki/MagicSquare.html.

Weisstein, E. W. "Books about Magic Squares." http://www.ericweisstein.com/encyclopedias/books/MagicSquares.html.

Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers.* Middlesex, England: Penguin Books, p. 75, 1986.

LAST MODIFIED: April 4, 2003

CITE THIS AS:

Weisstein, Eric W. "Magic Square." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/MagicSquare.html